



J. Chynoweth sc.



# EUCLIDE'S ELEMENTS;

*Euclid's*

The whole

## FIFTEEN BOOKS,

compendiously Demonstrated:

WITH

ARCHIMEDES'S Theorems of the  
Sphere and Cylinder, Investigated by the  
*Method of Indivisibles.*

ALSO,

## EUCLIDE'S DATA,

and a brief

Treatise of REGULAR SOLIDS.

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By ISAAC BARROW, D.D. *late Master of  
Trinity College in Cambridge.*

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The whole carefully Corrected, and Illustrated  
with Copper Plates.

To which is now added an

## A P P E N D I X,

Containing,

The Nature, Construction, and Application of  
Logarithms.

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By J. BARROW, Author of *Navigatio Britannica*, &c.

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ECCLIDÆ ELEMENTS

THE FIRST BOOK

CONTAINING THE FIRST SIX PROPOSITIONS

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## To the R E A D E R.

**I**F you are desirous, Courteous Reader, to know what I have performed in this Edition of the ELEMENTS OF EUCLIDE, I shall here explain it to you in short, according to the Nature of the Work. I have endeavoured to attain two Ends chiefly; the first, to be very perspicuous, and at the same time so very brief, that the Book may not swell to such a Bulk, as may be troublesome to carry about one, in both which I think I have succeeded. Some of a brighter Genius, and endued with greater Skill, might have demonstrated most of these Propositions with more nicety, but perhaps none with more succinctness than I have; especially since I alter'd nothing in the Number and Order of the Author's Propositions; nor presum'd either to take the Liberty of rejecting, as less necessary, any of them, or of reducing some of the easier sort into the Rank of Axioms, as several have done; and among others, that most expert Geometrician A. Tacquetus C. (whom I the more willingly name, because I think it is but civil to acknowledge that I have imitated him in some Points) after whose most accurate Edition I had no Thoughts of attempting any thing of this Nature, 'till I considered that this most learned Man thought fit to publish only Eight of EUCLIDE's Books, which he took the pains to explain and embellish, having in a manner rejected and undervalued the other Seven, as less appertaining to the Elements of Geometry. But my Province was originally quite different, not that of writing the Elements of Geometry after what method soever I pleas'd, but of demonstrating, in as few Words as possible I could, the whole Works of EUCLIDE. As



## To the READER.

to Four of the Books, viz. the Seventh, Eighth, Ninth, and Tenth, although they do not so nearly appertain to the Elements of plain and solid Geometry, as the six precedent and the two subsequent, yet none of the more skilful Geometricians can be so ignorant as not to know that they are very useful for Geometrical Matters, not only by reason of the very near affinity that is between Arithmetick and Geometry, but also for the Knowledge of both commensurable and incommensurable Magnitudes, so exceeding necessary for the Doctrine of both plain and solid Figures. Now the noble Contemplation of the five regular Bodies that is contained in the three last Books, cannot, without great Injustice, be pretermitted, since that for the sake thereof our σοιχειωτής, being a Philosopher of the Platonic Sect, is said to have compos'd this universal System of Elements; as Proclus lib. 2. witnesseth in these Words, "Ὅθεν δὴ καὶ τῆς συμπάσης σοιχειώσεως τέλος προεσήσατο τὴν τῶν καλεσμένων πλατωνικῶν χημάτων σύστασιν. Besides, I easily persuaded my self to think, that it would not be unacceptable to any Lover of these Sciences to have in his Possession the whole Euclidean Work, as it is commonly cited and celebrated by all Men: Wherefore I resolved to omit no Book or Proposition of those that are found in P. Herigonius's Edition, whose Steps I was obliged closely to follow, by reason I took a Resolution to make use of most of the Schemes of the said Book, very well foreseeing that Time would not allow me to form new ones, though sometimes I chose rather to do it. For the same Reason I was willing to use for the most part EUCLIDE's own Demonstrations, having only express'd them in a more succinct Form, unless perhaps in the Second, Thirteenth, and very few in the Seventh, Eighth, and Ninth Books, in which it seem'd not worth my while to deviate in any Particular from him: Therefore I am not without



## TO the READER

without great hopes, that as to this Part I have in some measure satisfied both my own Intentions, and the Desire of the Studious. As for some certain Problems and Theorems that are added in the Scholions (or short Expositions) either appertaining (by reason of their frequent Use) to the Nature of these Elements, or conducing to the ready Demonstration of those Things that follow, or which imitate the Reasons of some principal Rules of Practical Geometry, reducing them to their original Fountains, these I say, will not, I hope, make the Book swell to a Size beyond the design'd Proportion.

The other Butt which I levell'd at, is to content the Desires of those who are delighted more with symbolical than verbal Demonstrations. In which Kind, whereas most among us are accusom'd to the Symbols of Gulielmus Oughtredus, I therefore thought best to make use, ~~the~~ the most part, of his. None hitherto (as I know of) has attempted to interpret and publish EUCLIDE after this manner, except P. Herigonius; whose Method (tho' indeed most excellent in many things, and very well accommodated for the particular purpose of that most ingenious Man) yet it seems in my Opinion to labour under a double Defect. First, in regard, that, altho' of two or more Propositions produced for the Proof of any one Problem or Theorem, the former do not always depend on the latter, yet it do not readily enough appear, either from the order of each, or by any other manner, when they agree together, and when not; wherefore, for want of the Conjunctions and Adjectives, ergo, rursus, &c. many difficulties and occasions of doubt often arise in reading, especially to those that are Novices. Besides it frequently happens, that the said Method cannot avoid superfluous Repititions, by which the Demonstrations are often-  
times

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times render'd tedious, and sometimes also more intricate; which Faults my Method easily removes by the arbitrary mixture of both Words and Signs: Therefore let what has been said, touching the Intention and Method of this little Work, suffice. As to the rest, whoever covets to please himself with what may be said, either in Praise of the Mathematicks in general, or of Geometry in particular, or touching the History of these Sciences, and consequently of EUCLIDE himself, (who digested those Elements) and others ~~of that kind~~ of that kind, may consult other Interpreters. Neither will I (as if I were afraid lest these my Endeavours may fall short of being satisfactory to all Persons) alledge as an Excuse (though I may very lawfully do it) the want of due time which ought to be employ'd in this Work, nor the Interruption occasioned by other Affairs, nor yet the want of requisite help for these Studies, nor several other things of the like Nature. But what I have here employ'd my Labour and Study in for the Use of the ingenious Reader, I wholly submit to his Censure and Judgment, to approve if useful, or reject if otherwise.

I. B.



Thus



## To the R E A D E R.

Thus far the Learned Author. — But as the work has been often printed since his death, and by that means several errors committed, I have, at the request of the Booksellers concerned in this treatise, and from a sincere respect and veneration for the memory of the deceased Author, carefully revised the whole performance; and flatter myself, from the great pains and care I have taken, that very few errors will be found in this Edition.

And as the wooden cuts in the former editions, were, by often printing, almost obliterated, their place is now supplied by figures engraven on Copper Plates, and proper care taken to correct the inaccuracies and errors committed by cutting them on Wood; which has given this edition great advantages over the former, both with regard to beauty and correctness.

In the Appendix which I have added to this work, I have endeavoured to render the construction and use of Logarithms as plain and easy as possible. And that nothing might be wanting for understanding the nature of those tables used in trigonometrical calculations, I have added an investigation of the several series invented by the illustrious Sir Isaac Newton, for finding the length of the circumference of the circle, in equal parts of the radius, also of the sine, tangent and secant of any arch, in the same parts; with the application of these series to the constructing the triangular canon, and the quadrature of the Circle. I have also shewn the manner of computing the artificial or logarithmic sines, tangents, and secants, from the length of the arch of the circle first given in equal parts of the radius, independant of the tables of logarithms, sines, tangents, and secants.

J. BARROW.



## The Explication of the Signs or Characters.

|                |           |                                                                                                                                                   |
|----------------|-----------|---------------------------------------------------------------------------------------------------------------------------------------------------|
| $=$            | Signifies | Equal.                                                                                                                                            |
| $>$            |           | Greater.                                                                                                                                          |
| $<$            |           | Lesser.                                                                                                                                           |
| $+$            |           | More, or to be added.                                                                                                                             |
| $-$            |           | Less, or to be subtracted.                                                                                                                        |
| $:$            |           | The Differences, or Excess; Also, that all the Quantities which follow, are to be subtracted, the Signs not being changed.                        |
| $\times$       |           | Multiplication, or the Drawing one side of a Rectangle into another.<br>The same is denoted by the Conjunction of Letters; as $AB = A \times B$ . |
| $::$           |           | Continued Proportion.                                                                                                                             |
| $\sqrt{\quad}$ |           | The Side or Root of a Square, or Cube, &c.                                                                                                        |
| $Q \& q$       |           | A Square.                                                                                                                                         |
| $C \& c$       |           | A Cube.                                                                                                                                           |
| $Q. Q.$        |           | The Ratio of a square Number to a square Number.                                                                                                  |

*Other Abbreviations of Words, where-ever they occur, the Reader will without trouble understand of himself; saving some few, which, being of less general use, we refer to be explained in their Places, most commonly at the beginning of each Book in which they are used.*

The



# The FIRST BOOK OF EUCLIDE'S ELEMENTS.

## Definitions.

- I.  Point is that which hath no parts.
- II. A Line is a longitude without latitude.
- III. The ends, or limits of a line are points.
- IV. A Right-line is that which lies equally betwixt its points.
- V. A Superficies is that which hath only longitude and latitude.
- VI. The extremes, or limits of a superficies are lines.
- VII. A Plain-superficies is that which lies equally betwixt its lines.
- VIII. A Plain-angle is the inclination of two lines the one to the other, the one touching the other in the same plain, yet not lying in the same strait-line.
- IX. And if the lines which contain the angle, be right-lines, it is called a right-lined angle.

A

X. When

Plate I.  
Fig. 1.

X. When a right-line CG, standing upon a right-line AB, makes the angles on either side thereof, CGA, CGB, equal one to the other, then both those equal angles are right-angles; and the right-line CG, which standeth on the other, is termed a Perpendicular to that (AB) where, on it standeth.

Note, *When several angles meet at the same point (as at G) each particular angle is described by three letters; whereof the middle letter sheweth the angular point, and the two other letters the lines that make that angle: As the angle which the right-lines CG, AG make at G, is called CGA, or AGC.*

XI. An Obtuse-angle is that which is greater than a right-angle; as DGB.

XII. An Acute-angle is that which is less than a right-angle; as DGA.

XIII. A Limit, or Term, is the end of any thing.

XIV. A Figure is that which is contained under one or more terms.

Fig. 2.

XV. A Circle is a plain figure contained under one line, which is called a circumference; unto which all lines, drawn from one point within the figure, and falling upon the circumference thereof, are equal the one to the other.

XVI. And that point is called the center of the circle.

XVII. A Diameter of a circle is a right-line drawn thro' the center thereof, and ending at the circumference on either side, dividing the circle into two equal parts.

XVIII. A Semicircle is a figure which is contained under the diameter and that part of the circumference which is cut off by the diameter.

*In the circle EABCD, E is the center, AC the diameter, ABC the semicircle.*

XIX. Right-lined figures are such as are contained under right-lines.

XX. Three-sided or trilateral figures are such as are contained under three right-lines.

XXI. Four-sided or quadrilateral figures are such as are contained under four right-lines.

XXII. Many-sided figures are such as are contained under more right-lines than four.

XXIII. Of trilateral figures, that is, an equilateral triangle, which hath three equal sides; as the triangle ABC. Fig. 3.

XXIV.



XXIV. An *Isofceles*, is a triangle which hath only two sides equal; as the triangle ABC. Plate I. *Fig. 4.*

XXV. A *Scalenum*, is a triangle whose three sides are all unequal; as ABC. *Fig. 5.*

XXVI. Of these trilateral figures, a right-angled triangle is that which hath one right-angle; as the triangle ABD. *Fig. 5.*

XXVII. An *Amblygonium*, or obtuse-angle triangle, that is which hath one angle obtuse; as ABC. *Fig. 5.*

XXVIII. An *Oxgonium*, or acute-angled triangle, is that which hath three acute angles; as ABC. *Fig. 4.*

An equiangular, or equal-angled figure is that whereof all the angles are equal. Two figures are equiangular, if the several angles of the one figure be equal to the several angles of the other. The same is to be understood of equilateral figures.

XXIX. Of *Quadrilateral*, or four-sided figures, a square is that whose sides are equal, and angles right; as ABCD. *Fig. 6.*

XXX. A Figure on the one part longer, or a long square, is that which hath right-angles, but not equal sides; as ABCD. *Fig. 7.*

XXXI. A *Rhombus*, or diamond-figure, is that which has four equal sides, but is not right-angled; as ABCD. *Fig. 8.*

XXXII. A *Rhomboides*, is that whose opposite sides, and opposite angles are equal; but has neither equal nor right angles; as ABCD. *Fig. 9.*

XXXIII. All other quadrilateral figures besides these are called *trapezia*, or tables; as GNDH. *Fig. 10.*

XXXIV. Parallel, or equi-distant right-lines are such, which being in the same superficies, if infinitely produced, would never meet as AD and BC. *Fig. 11.*

A. 2 XXXV.

## The first Book of

XXXV. A Parallelogram is a quadrilateral figure, whose opposite sides are parallel, or equi-distant ; as ABCD. Plate I. Fig. 7.

XXXVI. In a Parallelogram AGEL, Fig. 9. when a diameter AE, and two lines BK, CF, parallel to the sides, cutting the diameter in one and the same point D, are drawn, so that the Parallelogram be divided by them into four Parallelograms ; those two LD, DG, through which the diameter does not pass, are called complements ; and the other two CB, KF, through which the diameter passeth, the Parallelograms standing about the diameter.

A Problem is, when something is proposed to be done or effected.

A Theorem is, when something is proposed to be demonstrated.

A Corollary is a Consequence, or some consequent truth gained from a preceding demonstration.

A Lemma is the demonstration of some premise, whereby the proof of the thing in hand becomes the shorter.

### Postulates or Petitions.

1. **F**rom any given point to any other given point, to draw a right-line.
2. To produce a finite right-line, strait forward continually.
3. Upon any center, and at any distance, to describe a circle.

### Axioms.

1. **T**Hings equal to the same thing, are also equal one to the other.

As  $A=B=C$ . Therefore  $A=C$  ; or therefore all A, B, C, are equal the one to the other.

Note, When several quantities are joyned the one to the other continually with this mark  $=$ , the first quantity is by virtue of this axiom equal to the last, and every one to every one : In which case we often abstain from citing the axiom, for brevity's sake ; altho' the force of the consequence depends thereon.

2. If to equal things you add equal things, the wholes will be equal.
3. If from equal things you take away equal things, the things remaining will be equal.
4. If



4. If to unequal things you add equal things, the wholes will be unequal.

5. If from unequal things you take away equal things, the remainders will be unequal.

6. Things which are double to the same third, or to equal things, are equal one to the other. Understand the same of triple, quadruple, &c.

7. Things which are half of one and the same thing, or of things equal, are equal the one to the other. Conceive the same of subtriple, subquadruple, &c.

8. Things which agree together, are equal one to the other.

*The converse of this axiom is true in right-lines and angles, but not in figures, unless they be like.*

*Moreover, magnitudes are said to agree, when the parts of the one being apply'd to the parts of the other, they fill up an equal or the same place.*

9. Every whole is greater than its part.

10. Two right-lines cannot have one and the same segment (or part) common to them both.

11. Two right-lines meeting in the same point, if they be both produced, they shall necessarily cut one another in that point.

12. All right-angles are equal the one to the other.

13. If a right-line EF (*Plate 1. Fig. 11.*) falling on two right-lines, AD, BC, make the internal angles on the same side, GFE, FEG, less than two right-angles, those two right-lines produced shall meet on that side where the angles are less than two right-angles.

14. Two right-lines do not contain a space.

15. If to equal things you add things unequal, the excess of the wholes shall be equal to the excess of the additions.

16. If to unequal things equal be added, the excess of the wholes shall be equal to the excess of those which were at first.

17. If from equal things unequal things be taken away, the excess of the remainders shall be equal to the excess of what was taken away.

18. If from things unequal things equal be taken away, the excess of the remainders shall be equal to the excess of the wholes.

19. Every whole is equal to all its parts taken together.



20. If one whole be double to another, and that which is taken away from the first be double to that which is taken away from the second, the remainder of the first shall be double to the remainder of the second.

The Citations are to be understood in this manner; When you meet with two numbers, the first shews the Proposition, the second the Book; as by 4. 1. you are to understand the fourth Proposition of the first Book; and so of the rest.

Moreover, ax. denotes Axiom, post. Postulate, def. Definition, sch. Scholium, cor. Corollary.

### PROPOSITION I. Plate I. Fig. 12.

**U**Pon a finite right-line given as AB, to describe an equilateral triangle ACB.

From the centers A and B, at the distance of AB, or BA, (a) describe two circles cutting each other in the point C; from whence (b) draw two right-lines CA, CB. Then is AC (c) = AB (c) = BC (d) = AC. (e) Wherefore the triangle ACB is equilateral. Which was to be done.

a 3. post.  
b 1. post.  
c 15. def.  
d 1. ax.  
e 23. def.

#### Scholium.

After the same manner upon the line AB may be described an Ilosceles triangle, if the distances of the equal circles be taken greater or less than the line AB.

### PROP. II. Fig. 13.

From a point given A, to draw a right-line AG equal to a right-line given BC.

From the center C, at the distance of CB, (a) describe the circle CBE. (b) Join AC; upon which (c) raise the equilateral triangle ADC. (d) Produce DC to E. From the center D, at the distance of DE, describe the circle DEG; and let DA (e) be produced to the point G in the circumference thereof. Then AG = CB.

For DG (f) = DE, and DA (g) = DC. Wherefore AG (h) = CE (i) = BC (l). Which was to be done.

The putting of the point A within or without the line BC varies the cases; but the construction, and the demonstration, are every where alike.

a 3. post.  
b 1. post.  
c 1. 1.  
d 2. post.  
e 2. post.

f 15. def.  
g constr.  
h 3 ax.  
k 15. def.  
l 1. ax.

Schol.

Schol.

The line AG might be taken with a pair of compasses; but the so doing answers to no postulate, as Proclus well observes.

PROP. III. Plate I. Fig. 14.

Two right-lines, A and DC, being given, from the greater DC, to take away the right-line DE, equal to the lesser A.

From the point D (a) draw the right-line BD = A. The circle described from the center D at the distance of BD shall cut off DE (b) = BD (c) = A (d) = DE. Which was to be done.

a 2. 1.  
b 15. def.  
c constr.  
d 1. ax.

PROP. IV. Fig. 15, and 16.

If two triangles BAC, EDF, have two sides of the one BA, AC, equal to two sides of the other ED, DF, each to its correspondent side (that is  $BA = ED$ , and  $AC = DF$ ) and have the angel A equal to the angle D contained under the equal right-lines; they shall have the base BC equal to the base EF; and the triangle BAC shall be equal to the triangle EDF; and the remaining angles B, C, shall be equal to the remaining angles, E, F, each to each, under which the equal sides are subtended.

If the point D be apply'd to the point A, and the right-line DE plac'd upon the right-line AB, the point E shall fall upon B, because  $DE (a) = AB$ , also the right-line DF shall fall upon AC, because the angle A (a) = D. Moreover the point F shall fall upon the point C, because  $AC (a) = DF$ . Therefore the right-lines EF, BC, shall agree, because they have the same terms, and consequently are equal. Wherefore the triangles, BAC, DEF, and the angles B, E, as also the angles C, F, do agree, and are equal. Which was to be demonstrated.

a hyp.

PROP. V. Fig. 4.

The angles BAC, BCA, at the base of an Isosceles triangle ABC, are equal one to the other, And if the equal sides AB, BC, are produced, the angles CAD, ACE, under the base, shall be equal one to the other.

A 4

(a) Take



a 3. 1.  
b 1. post.  
c hyp.  
d constr.  
e 4. 1.  
f 3. ax.  
g 4. 1.  
h before.  
k 3. ax.

(a) Take  $BE = BD$ ; and (b) join  $CD$ , and  $AE$ .  
Because, in the triangles  $BCD$ ,  $BAE$ , are  $AB$  (c)  $=$   $BC$ , and  $BE$ , (d)  $=$   $BD$ , and the angle  $B$  common to them both, (e) therefore is the angle  $BAE = BCD$ , and the angle  $AEB$  (e)  $=$   $BDC$ , and the base  $AE$  (e)  $=$   $CD$ ; also  $EC$  (f)  $=$   $DA$ . Therefore in the triangles  $AEC$ ,  $ADC$  (g) will be the angle  $ECA = DAC$ . Which was to be dem. Also the angle  $EAC = DCA$ , but the angle  $BAE$  (b)  $=$   $BCD$ ; therefore the angle  $BAC$  (k)  $=$   $BCA$ . Which was to be demonstrated.

Coroll.

Hence, every equilateral triangle is also equiangular.

P R O P. VI. Plate I. Fig. 4.

If two angles  $BCA$ ,  $ACB$  of a triangle  $ABC$ , be equal the one to the other, the sides  $BC$ ,  $AB$ , subtended under the equal angles, shall also be equal one to the other.

If the sides be not equal, let one be bigger than the other, suppose  $BA > BC$ . (a) Make  $AF = BC$ , and (b) draw the line  $CF$ .

In the triangles  $FAC$ ,  $BAC$ , because  $AF$  (c)  $=$   $CB$ , and the side  $AC$  is common, and the angle  $BAC$  (d)  $=$   $ACB$ , the triangles  $FAC$ ,  $ACB$  (e) shall be equal the one to the other, a part to the whole, (f) Which is impossible,

Coroll.

Hence, every equiangular triangle is also equilateral.

P R O P. VII. Fig. 4.

Upon the same right-line  $AC$  two right-lines being drawn  $AB$ ,  $BC$ , two other right-lines equal to the former,  $AF$ ,  $CF$ , each to each (viz.  $AF = AB$ , and  $CF = BC$ ) cannot be drawn from the same points  $A$ ,  $C$ , on the same side  $B$ , to several points, as  $B$  and  $F$ , but only to  $B$ .

1. Case. If the point  $F$  be set in the line  $AB$ , it is plain that  $AF$  is (a) not equal to  $AB$ .

2. Case. If the point  $D$  be placed within the triangle  $ACB$ , then draw the line  $CD$ , and produce  $BDF$ , and  $BCE$ . Now you would have  $AD = AC$ , then the angle  $ADC$  (b)  $=$   $ACD$ ; as also, because  $BD$  (c)  $=$   $BC$ , the angle  $FDC$   $=$   $FCD$  (d)

a 9 ax.

Fig. 17.

b 5. 1.

c suppos.



(b) ECD, therefore is the angle FDC — (d) ACD, that is, the angle FDC — ADC. (d) Which is impossible.

d 9 ax.

3. Case. If D (Fig. 18.) falls without the triangle ACB, let CD be joined.

Again, the angle ACD (e) = ADC, and the angle BCD (e) = BDC. (f) Therefore the angle ACD — BDC, viz. the angle ADC — BDC. Which is impossible. Therefore, &c.

e 5. 1.

f 9. ax.

PROP. VIII. Plate I. Fig. 15, and 16.

If two triangles ABC, DEF have two sides AB, AC, equal to two sides DE, DF, each to each, and the base BC equal to the base EF, then the angles contained under the equal right-lines shall be equal, viz. A to D.

Because, BC (a) = EF, if the base BC be laid on the base EF, (b) they will agree: therefore whereas AB (c) = DE, and AC = DF, the point A will fall on D (for it cannot fall on any other point, by the precedent proposition) and so the sides of the angles A and D are coincident; (d) wherefore those angles are equal. Which was to be demonstrated.

a hyp.

b ax. 8.

c hyp.

d 8. ax.

Coroll.

1. Hence, triangles mutually equilateral are also mutually (x) equiangular.

2. Triangles mutually equilateral (x) are equal one to the other.

x 4. 1.

PROP. IX. Fig. 19.

To bisect, or divide into two equal parts, a right-lined angle given BAC.

(a) Take AD = AE, and draw the line DE; upon which (b) make an equilateral triangle DFE, draw the right-line AF; it shall bisect the Angle.

a 3. 1.

b 1. 1.

For AD (c) = AE, and the side AF is common, and the base DF (c) = FE. (d) therefore the angle DAF = EAF. Which was to be done.

c constr.

d 8. 1.

Coroll.

Hence it appears how an angle may be cut into 4, 8, 16, 32, &c. equal parts, to wit, by bisecting each part again.

The method of cutting angles into any number of equal parts required, by a Rule and Compass, is as yet unknown to Geometricians.

PROP.

## PROP. X. Plate I. Fig. 20.

To bisect a right-line given AB.

a 1. 1.

b 9. 1.

c constr.

d 4. 1.

Upon the line given AB (a) erect an equilateral triangle ABC; and (b) bisect the angle C with the right-line CD. That line shall also bisect the line given AB.

For AC (c) = BC, and the side CD is common, and the angle ACD (c) = BCD. therefore AD (d) = BD. Which was to be done.

The practice of this and the preceding proposition is easily shewn by the Construction of the 1st proposition of this Book.

## PROP. XI. Fig. 20.

From a point D in a right-line given AB to erect a right-line DC at right-angles.

a 3. 1.

b 1. 1.

c constr.

d 8. 1.

e 10. def.

(a) Take on either side of the point given DA = DB, upon the right-line AB (b) erect an equilateral triangle, draw the line CD, and it will be the perpendicular required.

For the triangles DAC, DCB are mutually (c) equilateral, (d) therefore the angle ADC = CDB; (e) therefore DC is perpendicular. Which was to be done.

The practice of this and the following is easily performed by the help of a square.

## PROP. XII. Fig. 20.

Upon an infinite right-line given AB, from a point given that is not in it, to let fall a perpendicular right-line CD.

a 3 post.

b 10. 1.

c constr.

d 8. 1.

e 10. def.

From the center C (a) describe a circle cutting the right-line given in the points A and B. Then (b) bisect AB in D, and draw the right-line in CD, which will be the perpendicular required.

Let the lines CA, CB be drawn. The triangles ADC CDB are mutually (c) equilateral; (d) therefore the angles ADC, BDC are equal, and by (e) consequence right. (e) Wherefore DC is a perpendicular. Which was to be done.

## PROP. XIII. Fig. 1.

When a right-line DG standing upon a right-line AB maketh angles DGA, DGB; it maketh either two right-angles, or two angles equal to two right-angles.

¶



If the angles DGA, DGB be equal, (a) then they make two right-angles; if unequal, then from the point G (b) let there be erected a perpendicular GC. Because the angle DGB (c) = to a right-angle + CGD, and the angle DGA (d) = to a right-angle - DGC; therefore the angle DGB - DGA (e) = to two right-angles. Which was to be demonstrated.

a def. 10.

b 11. 1.

c 19. ax.

d 3. ax.

e 2. ax.

Corollaries.

1. Hence, if one angle DGB be right, the other DGA is also right; if one acute, the other is obtuse, and so on the contrary.

2. If more right-lines than one stand upon the same right-line at the same point, the angles shall be equal to two right.

3. Two right-lines cutting each other make angles equal to four right-angles.

4. All the angles made about one point make four right-angles; as appears by Coroll. 2.

P R O P. XIV. Plate I. Fig. 1.

If to any right-line CG, and a point therein G two right-lines, not drawn from the same side, do make the angles CGA, CGB, on each side equal to two right, the lines, AG, GB, shall make one strait-line.

If you deny it, let AG, GF make one right-line; then shall the angle CGA + CGF (a) = two right angles (b) = CGA + CGB. Which is (c) absurd.

a 13. 1.

b hyp.

c 9. ax.

P R O P. XV. Fig. 21.

If two right-lines AB, CD, cut thro' one another, then are the two angles which are opposite, viz. CEB, AED, equal one to the other.

For the angle AEC + CEB (a) = to two right angles = AEC + AED; (b) therefore CEB = AED. Which was to be demonstrated.

a 13. 1.

b 3. ax.

Schol. 1.

If to any right-line AB, and in it a point E, two right lines being drawn CE, ED, and not taken on the same side, make the vertical (or opposite) angles CEA, and BED equal, those right-lines CE, ED, do meet directly and make one strait line.

For



a 13. 1.

b 2. ax.

c 14. 1.

For two right-angles are (a) equal to the angle  $CEA + CEB$ , (b)  $= CEB + BED$ . (c) Therefore  $CE, ED$ , are in a strait line. *Which was to be demonstrated.*

Schol. 2. Plate I. Fig. 21.

If four right-lines  $EA, EB, EC, ED$ , proceeding from one point  $E$ , make the angles, vertically opposite, equal the one to the other, each two lines,  $AE, EB$ , and  $CE, ED$ , are placed in one strait line.

a 4. c 13. 1.

b hyp.  $\mathcal{E}$ 

c 2. ax.

c 14. 1.

For because the angle  $AEC + AED + CEB + DEB$  (a)  $=$  to four right-angles, therefore the angle  $AEC + AED$  (b)  $= CEB + DEB =$  to two right-angles. (c) Therefore  $CED$  and  $AEB$  are strait-lines. *Which was to be demonstrated.*

P R O P. XVI. Fig. 22.

One side,  $BC$ , of any triangle  $ABC$  being produced, the outward angle  $ACD$  will be greater than either of the inward and opposite angles,  $CAB, CBA$ .

a 10. 1.  $\mathcal{E}$ 

1. post.

b 3. 1.

Let the right-lines  $AH, BE$ , (a) bisect the sides  $AC, BC$ ; from which lines produc'd, take (b)  $EF = BE$ , and  $HI, (b) = AH$ , and join  $FC$ , and  $IC$ ; and produce  $ACG$ .

c constr.

d 15. 1.

e 4. 1.

f 15. 1.

g 9. ax.

Because  $CE (c) = EA$ , and  $EF (c) = EB$ , and the angle  $FEC (d) = BEA$ , the angle  $ECF (e)$  shall be equal to  $EAB$ . By the like argument is the angle  $ICH = ABH$ . Therefore the whole angle  $ACD (f)$  ( $BCG$ ,) (g) is greater than either the angle  $CAB$  or  $ABC$ . *Which was to be demonstrated.*

P R O P. XVII. Fig. 23.

Two angles of any triangle  $ABC$ , which way soever they are taken, are less than two right-angles.

a 13. 1.

b 16. 1.

c 4. ax.

Let the side  $BC$  be produced. Because the angle  $ACD + ACB (a) =$  two right-angles, and the angle  $ACD (b)  $\square$   $A$ , (c) therefore  $A + ACB  $\sqsubset$$  than two right-angles. After the same manner is the angle  $B + ACB  $\sqsubset$$  than two right-angles. Lastly, the side  $AB$  being produced, the angle  $A + B$  will be also less than two right-angles. *Which was to be demonstrated.*$

Coroll.

Coroll.

1. Hence it follows that in every triangle wherein one angle is either right or obtuse, the two others are acute angles.

2. If a right-line AE make unequal angles with another right-line BC, one acute AEB, the other obtuse AEC, a perpendicular AB, let fall from any point A to the other line BC, shall fall on that side the acute angle is of.

For if AC, drawn on the side of the obtuse-angle, be a perpendicular, then in the triangle AEC, shall the angles AEC + ACE be greater than two right-angles.

\* Which is contrary to the precedent prop.

3. All the angles of an equilateral triangle, and the two angles of an Isosceles triangle that are upon the base, are acute.

P R O P. XVIII. Plate I. Fig. 23.

The greatest side AC of every triangle ABC subtends the greatest angle ABC.

From AC (a) take away AF = AB, and join BF. (b) Therefore is the angle AFB = ABF. But AFB (c)  $\square$  C; therefore is ABF  $\square$  C; (d) therefore the whole angle ABC  $\square$  C. After the same manner shall be ABC  $\square$  A. Which was to be demonstrated.

P R O P. XIX. Fig. 23.

In every triangle ABC, under the greatest angle B is subtended the greatest side AC.

For if AB be supposed equal to AC, then will be the angle B (a) = C, which is contrary to the Hypothesis: And if AB  $\square$  AC, then shall be the angle C (b)  $\square$  B, which is against the Hypothesis. Wherefore rather AC  $\square$  AB; and after the same manner AC  $\square$  BC. Which was to be demonstrated.

P R O P. XX. Fig. 24.

Of every triangle ABC, two sides BA, AC, any way taken, are greater than the side that remains BC.

Pro-

\* 17. 1.

a 3. 1.  
b 5. 1.  
c 16. 1.  
d 9. ax.

a 5. 1.

b 18. 1.



- a 3. 1.  
b 5. 1.  
c 9. ax.  
d 19. 1.  
e constr.  
E 2. ax.

Produce the line BA, (a) and take  $AD=AC$ , and draw the line DC; (b) then shall the angle D be equal to ACD; (c) therefore is the whole angle BCD  $\square$  D; (d) therefore BD (e)  $(BA+AC) \square BC$ . Which was to be demonstrated.

PROP. XXI. Plate I. Fig. 25.

If from the utmost points of one side BC, of a triangle ABC, two right-lines BD, CD, be drawn to any point within the triangle, then are both those two lines shorter than the two other sides of the triangle BA, CA; but contain a greater angle, BDC.

- a 20. 1.  
b 4. ax.

Let BD be produced to E. Then is  $CE+ED$  (a)  $\square$  CD, and BD common to both, (b) then shall be  $DB+DE+EC \square CD+BD$ . Again,  $BA+AE$  (a)  $\square$  BE; (b) therefore  $BA+AC \square BE+EC$ . Wherefore 1.  $BA+AC \square BD+DC$ . 2. The angle BDC (c)  $\square$  DEC (c)  $\square$  A. Therefore the angle BDC  $\square$  A. Which was to be demonstrated.

- c 16. 1.

PROP. XXII. Fig. 26.

To make a triangle FKG of three right-lines FK, FG, GK, which shall be equal to three right-lines given A, B, C. Of which it is necessary that any two taken together be longer than the third.

- a 3. 1.  
b 3. post.

From the infinite line DE (a) take DF, FG, GH, equal to the lines given A, B, C. Then if from the (b) centers F and G at the distances of FD and GH, two circles be drawn cutting each other in K, and the right-lines KF, KG be joined, the triangle FKG shall be made, (c) whose sides FK, FG, GK, are equal to the three lines DF, FG, GH, (d) that is, to the three lines given A, B, C. Which was to be done.

- c 15. def.

- d 1. ax.

PROP. XXIII. Fig. 27, and 28.

At a point A in a right-line given AH, to make a right-lined angle A equal to a right-lined angle given D.

- a 1. post.  
b 3. 1.  
c 22. 1.  
d 8. 1.

(a) Draw the right-line CF cutting the sides of the angle given any ways; (b) make  $AG=CD$ ; upon AG (c) raise a triangle equilateral to the former CDF, so that AH be equal to DF, and GH to CF. then shall you have the angle A (d)  $=D$ . Which was to be done.

PROP.

PROP. XXIV. Plate I. Fig. 29, 30.

If two triangles ABC, DEF have two sides of the one triangle AB, AC, equal to two sides of the other triangle DE, DF, each to each, and have the angle A greater than the angle EDF contained under the equal right-lines, they shall have also the Base BC greater than the base EF.

(a) Let the angle EDG be made equal to A, and the side DG (b) = DF (c) = AC; and let EG, and FG be joined.

1. Case. If EG falls above EF; Because AB (d) = DE, and AC (e) = DG, and the angle A (e) = EDG; (f) therefore is BC = EG. But because DF (c) = DG; (g) therefore is the angle DFG = DGF; (h) therefore is the angle DFG = EGF, and by consequence the angle EFG, (i) = EGF; (k) wherefore EG (BC) = EF.

2. Case. If the base EF coincides with the base EG, (l) it is evident that EG (BC) = EF. Fig. 29, 31.

3. Case. If EG falls below EF, then because DG + GE (m) = DF + FE, if from both be taken away DG, DF which are equal; EG (BC) remains (n) = EF. Which was to be demonstrated. Fig. 29, 32.

a 23. 1.

b 3. 1.

c hyp.

d hyp.

e constr.

f 4. 1.

g 5. 1.

h 9. ax.

k 19. 1.

l 9. ax.

m 21. 1.

n 5. ax.

PROP. XXV. Fig. 29, 30.

If two triangles ABC, DEF, have two sides AB, AC, equal to two sides DE, DF, each to each, and have the base BC greater than the base EF, they shall also have the angle A contained under the equal right-lines greater than the angle EDF.

For if the angle A be said to be equal to EDF, (a) then is the base BC = EF, which is against the Hypothesis. If it be said the angle A > EDF, then (b) will be BC > EF, which is also against the Hypothesis. Therefore A = EDF. Which was to be demonstrated.

a 4. 1.

b 24. 1.

PROP. XXVI. Fig. 29, 31.

If two triangles BAC, EDG, have two angles of the one B, C, equal to two angles of the other E, DGE, each to his correspondent angle, and have also one side of the one equal to one side of the other, either that side which lyeth betwixt the equal angles, or that which is subtended under one of the equal angles; the other sides also of the one shall be equal to the



the other sides of the other, each to his correspondent side, and the other angle of the one, shall be equal to the other angle of the other.

1. Hypothesis. Let  $BC$  be equal to  $EG$ , which are the sides that lie between the equal angles, Then I say  $BA = ED$ , and  $AC = DG$ , and the angle  $A = EDG$ . For if it be said that  $ED < BA$ , then (a) let  $EH$  be made equal to  $BA$ , and let the line  $GH$  be drawn.

Because  $AB(b) = HE$ , and  $BC(c) = EG$ , and the angle  $B(c) = E$ , therefore shall be the angle  $EGH(d) = C(e) = DGE$ . (f) Which is absurd, therefore  $AB = ED$ . After the same manner  $AC$  may be proved equal to  $DG$ , (d) then will the angle  $A$  be equal to  $EDG$ .

2. Hyp. Let  $AB$  be equal to  $DE$ , then I say  $BC = EG$ , and  $AC = DG$ , and the angle  $A = EDG$ . For if  $EG$  be greater than  $BC$  make  $EF = BC$ , and join  $DF$ . Now because  $AB(g) = DE$ , and  $BC(b) = EF$ , and the angle  $B(g) = E$ ; therefore will be the angle  $EFD(h) = C(i) = EGD$ . (m) Which is absurd. Therefore is  $BC = EG$ , and so as before,  $AC = DG$ , and the angle  $A = EDG$ . Which was to be dem.

P R O P. XXVII. Plate I. Fig. 33.

If a right-line  $EF$ ; falling upon two right-lines  $AB, CD$ , makes the alternate angles  $A b F, D c E$ , equal the one to the other, then are the right-lines  $AB, CD$ , parallel.

If  $AB, CD$  be said not to be parallel, produce them till they meet in  $O$ , which being supposed, the outward angle  $A b F$  will be (a) greater than the inward angle  $D c E$ , to which it was equal by Hypothesis. Which things are repugnant.

P R O P. XXVIII. Fig. 33.

If a right-line  $EF$ , falling upon two right-lines,  $AB, CD$ , makes the outward angle  $A G E$  of the one line equal to  $C H G$  the inward and opposite angle of the other on the same side, or make the inward angles on the same side  $A b H, C c G$ , equal to two right-angles, then are the right-lines,  $AB, CD$ , parallel.

Hyp. 1. Because by Hypothesis the angle  $A b E = C c G$ , (a) therefore are  $B b H, C c G$ , the alternate angles equal; And (b) therefore are  $AB$  and  $CD$  parallel. Hyp.

a 3. 1.

b constr.

c hyp.

d 4. 1.

e hyp.

f 9. ax.

g hyp.

h constr.

k 4. 1.

l hyp.

m 16. 1.

a 16. 1.

a 15. 1.

b. 27. 1.

Hyp. 2. Because by Hypothesis the angle  $AGH + CHG =$  to two right-angles,  $(a) = AGH + BGH$ ,  $(b)$  therefore shall the angle  $CHG = BGH$ ; and  $(c)$  therefore  $AB, CD$ , are parallel, Which was to be demonstrated.

a 13. 1.  
b 3. ax.  
c 27. 1.

P R O P. XXIX. Plate I. Fig. 33.

If a right-line  $EF$  falls upon two parallels,  $AB, CD$ , it will make both the alternate angles  $DHG, AGH$ , equal each to the other, and the outward angle  $BGE$  equal to the inward and opposite angle on the same side  $DHG$ , as also the inward angles on the same side  $AGH, CHG$ , equal to two right-angles.

It is evident, that  $AGH + CHG =$  two right-angles;  $(a)$  otherwise  $AB, CD$ , would not be parallel, which is contrary to the Hypothesis: But moreover the angle  $DHG + CHG$   $(b) =$  two right-angles; therefore is  $DHG$   $(c) = AGH$   $(d) = BGE$ . Which was to be dem.

a 13. ax.  
b 13. 1.  
c 3. ax.  
d 15. 1.

Coroll. Fig. 7.

Hence it follows that every parallelogram  $AD$  having one angle right  $A$ , the rest are also right.

For  $A + B$   $(a) =$  two right-angles. Therefore, where as  $A$  is right,  $(b)$   $B$  must be also right. By the same argument are  $C$  and  $D$  right-angles.

a 29. 1.  
b 3. ax.

P R O P. XXX. Fig. 21.

Right-lines  $(AB, MN)$  parallel to one and the same right-line  $KL$ , are also parallel the one to the other.

Let  $CD$  cut the three right-lines given any ways. Then because  $AB, MN$  are parallel, the angle  $BEI$  will be  $(a) = LHI$ . Also because  $KL$  and  $MN$  are parallel, the angle  $LHI$  will be  $(a) = MIC$   $(b)$  Therefore the angle  $BEI = MIC$ ;  $(c)$  whence  $AB$  and  $MN$  are parallel. Which was to be demonstrated.

a 29. 1.  
a 29. 1.  
b 1. ax.  
c 27. 1.

P R O P. XXXI. Fig. 34.

From a point given  $A$ , to draw a right-line  $AE$ , parallel to a right-line given  $BC$ .

From the point  $A$  draw a right-line  $AD$  to any point of the given right-line; with which at the point thereof  $(a)$   $A$  make an angle  $DAE = ADC$ ;  $(b)$  then will  $AE$  and  $BC$  be parallel. Which was to be done.

a 23. 1.  
b 27. 1.

B

P R O P.



## PROP. XXXII. Plate I. Fig. 35.

Of any triangle ABC one side BC being drawn out, the outward angle ACD shall be equal to the two inward opposite angles A, B, and the three inward angles of the triangle, A, B, ACB, shall be equal to two right-angles.

a 31. 1.  
b 29. 1.  
c 2 ax.  
d 19. ax.  
e 13. 1.  
f 1. ax.

From C (a) draw CE parallel to BA. Then is the angle A (b) = ACE, and the angle B (b) = ECD. Therefore  $A + B$  (c) =  $ACE + ECD$  (d) = ACD. Which was to be demonstrated.

I affirm  $ACD + ACB$  (e) = two right-angles; (f) therefore  $A + B + ACB$  = two right-angles. Which was to be demonstrated.

## Coroll.

1. The three angles of any triangle taken together are equal to the three angles of any other triangle taken together. From whence it follows,

2. That if in one triangle, two angles (taken severally, or together) be equal to two angles of another triangle (taken severally, or together) then is the remaining angle of the one equal to the remaining angle of the other. In like manner, if two triangles have one angle of the one equal to one of the other, then is the sum of the remaining angles of the one triangle equal to the sum of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equal to a right-angle. Likewise, that angle in a triangle which is equal to the other two, is it self a right-angle.

4. When in an Isosceles the angle made by the equal sides is right, the other two upon the base are each of them half a right-angle.

5. An angle of an equilateral triangle makes two thirds parts of a right-angle. For one third of two right-angles is equal to two thirds of one.

## Schol.

By the help of this proposition you may know how many right-angles the inward and outward angles of a right-lined figure make; as may appear by these two following Theorems.

THE O

THEOREM I.

*All the angles of a right-lined figure do together make twice as many right-angles, abating four, as there are sides of the figure.*

From any point within the figure let right-lines be drawn to all the angles of the figure, which shall resolve the figure into as many triangles as there are sides of the figure. Wherefore, whereas every triangle affords two right-angles, all the triangles taken together will make up twice as many right-angles as there are sides. But the angles about the said point within the figure make up four right; therefore, if from the angles of all the triangles you take away the angles which are about the said point, the remaining angles, which make up the angles of the figure, will make twice as many right-angles, abating four, as there are sides of the figure. *Which was to be demonstrated.*

*Coroll.*

Hence all right-lined figures of the same species have the sums of their angles equal.

THEOREM II.

*All the outward angles of any right-lined figure, taken together, make up four right-angles.*

For every inward angle of a figure, with the outward angle of the same, make two right-angles; therefore all the inward angles, together with all the outward, make twice as many right-angles as there are sides of the figure; but (as has been just shewn) all the inward angles, with four right, make twice as many right as there as sides of the figure, therefore the outward angles are equal to four right-angles. *Which was to be demonstrated.*

*Coroll.*

All right-lined figures, of whatsoever species, have the sums of their outward angles equal.

B z

P R O P.



## PROP. XXXIII. Plate I. Fig. 7.

If two equal and parallel lines  $AB, CD$ , be joyned together with two other right-lines,  $AC, BD$ , then are those lines also equal and parallel.

a 29. 1.

b 4. 1.

c 27. 1.

Draw a line from  $C$  to  $B$ . Now because  $AB$  and  $CD$  are parallel, and the angle  $ABC$  ( $a$ )  $= BCD$ ; and also by hypothesis  $AB = CD$ , and the side  $CB$  common, therefore is  $AC$  ( $b$ )  $= BD$ , and the angle  $ACB$  ( $b$ )  $= DBC$  ( $c$ ) whence also  $AC, BD$ , are parallel,

## PROP. XXXIV. Fig. 7.

In parallelograms, as  $ABDC$ , the opposite sides  $AB, CD$ , and  $AC, BD$ , are equal each to the other; and the opposite angles  $A, D$ , and  $ABD, ACD$ , are also equal; and the diameter  $BC$  bisects the same.

a hyp.

b 29. 1.

c 2. ax.

d 26. 1.

Because  $AB, CD$ , ( $a$ ) are parallel, ( $b$ ) therefore is the angle  $ABC = BCD$ . Also because  $AC, BD$ , are ( $a$ ) parallel, ( $b$ ) therefore is the angle  $ACB = CBD$ ; ( $c$ ) therefore the whole angle  $ACD = ABD$ . After the same manner is  $A = D$ . Moreover because the angles  $ABC, ACB$ , lie at each end of the side  $CB$ , and are equal to  $BCD, CBD$ , ( $d$ ) therefore is  $AC = BD$ , and  $AB$  ( $d$ )  $= CD$ , and so the triangle  $ABC = CBD$ . Which was to be demonstrated.

Schol.

Every four-sided figure  $ABDC$ , having the opposite sides equal, is a parallelogram.

a 27. 1.

b 35. def. 1.

For by 8. 1. the angle  $ABC = BCD$ ; ( $a$ ) wherefore  $AB, CD$ , are parallel. In like manner is the angle  $BCA = CBD$ ; ( $a$ ) wherefore  $AC, BD$ , are also parallel. ( $b$ ) Therefore  $ABCD$  is a parallelogram. Which was to be demonstrated.

Fig. 36.

From hence we may more expeditiously draw a parallel  $CG$  to a right-line given,  $AB$ , thro' a point assigned,  $C$ .

Take in the line  $AB$  any point, as  $E$ . From the centers  $E$  and  $C$  at any distance draw two equal circles  $EF, CD$ . From the center  $F$  with the distance  $EC$  draw a circle  $FD$ , which shall cut the former circle  $CD$  in the point  $D$ . Then shall the line drawn  $CG$  be parallel to  $AB$ , for, as it was before demonstrated  $CEFD$  is a parallelogram.

PROP

PROP. XXXV. Plate I. Fig. 37.

Parallelograms,  $BCDA$ ,  $BCFE$ , which stand upon the same base  $BC$ , and between the same parallels  $AF$ ,  $BC$ , are equal one to the other.

For  $AD$  ( $a$ ) =  $BC$  ( $a$ ) =  $EF$ , add  $DE$  common to both; ( $b$ ) then is  $AE$  =  $DF$ . But also  $AB$  ( $a$ ) =  $DC$ , and the angle  $A$  ( $c$ ) =  $CDF$ . ( $d$ ) Therefore is the triangle  $ABE$  =  $DCF$ . Take away  $DgE$  common to both triangles; ( $e$ ) then is the Trapezium  $ABgD$  =  $EgCF$ ; add  $BgC$ , common to both; ( $f$ ) then is the parallelogram  $ABCD$  =  $EBCF$ . Which was to be demonstrated.

a 34. 1.  
b 2. ax.  
c 29. 1.  
d 4. 1.  
e 3. ax.  
f 2. ax.

The demonstration of any other cases, is not unlike, but much more plain and easy.

Schol. Fig. 7.

If the side  $AB$ , of a right-angled parallelogram  $ABCD$ , be conceived to be carried along perpendicularly thro' the whole line  $AC$ , or  $AC$  thro' the whole line  $AB$ , the area or content of the rectangle  $ABCD$  shall be produc'd by that motion. Hence a rectangle is said to be made by the drawing or multiplication of two contiguous sides. For example; let  $AB$  be supposed four foot, and  $AC$  three; draw three into four, there will be produced twelve square feet for the area of the rectangle.

This being supposed, the dimension of any parallelogram ( $EBCF$ ) is found out by this theorem. For the area thereof is produced from the altitude  $BA$  drawn into the base  $BC$ . For the area of the rectangle  $AC$  = parallelogram  $EBCF$ , is made by the drawing of  $BA$  into  $BC$ , therefore, &c.

Fig. 37.

PROP. XXXVI. Fig 37.

Parallelograms  $BCDA$ ,  $GHFE$ , standing upon equal bases  $BC$ ,  $GH$ , and betwixt the same parallels  $AF$ ,  $BH$ , are equal one to the other.

Draw  $BE$  and  $CF$ . Because  $BC$  ( $a$ ) =  $GH$  ( $b$ ) =  $EF$ , ( $c$ ) therefore is  $BCFE$  a parallelogram, Whence the parallelogram  $BCDA$  ( $d$ ) =  $BCFE$  ( $d$ ) =  $GHFE$ . Which was to be demonstrated.

a hyp.  
b 34. 1.  
c 33. 1.  
d 35. 1.

B 3

PROP.



## PROP. XXXVII. Plate I. Fig. 38.

Triangles, BCA, BCD, standing upon the same base BC, and between the same parallels BC, EF, are equal the one to the other.

a 31. I.  
b 34. I.  
c 35. I.  
and 7. ax.

(a) Draw BE parallel to CA, (a) and CF parallel to BD. Then is the triangle BCA (b) = half Pgr. BCAF (c) = half BDFC (b) = BCD. Which was to be demonstrated.

## PROP. XXXVIII. Fig. 38.

Triangles, BCA, DFC, set upon equal bases BC, DF, and between the same parallels EF, BC, are equal the one to the other.

a 34. I.  
b 36. I.  
and 7. ax.  
c 34. I.

Draw EB parallel to AC, and DB parallel to FC. Then is the triangle BCA (a) = half Pgr. BCAF (b) = half BDFC (c) = CDF. Which was to be demonstrated.

Schol.

If the base BC be greater than DF, then is the triangle BAC  $\supset$  DFC, and so on the contrary.

## PROP. XXXIX. Fig. 39.

Equal triangles BCA, BCD, standing on the same base BC, and on the same side are also between the same parallels AD, BC.

a 37. I.  
b hyp.  
c 9. ax.

If you deny it, let another line AF be parallel to BC; and let CF be drawn. Then is the triangle CBF (a) = CBA (b) = CBD. (c) Which is absurd.

## PROP. XL. Fig. 40.

Equal triangles BCA, CFD, standing upon equal bases BC, CF, and on the same side, are betwixt the same parallels.

a 38. I.  
b hyp.  
c 9. ax.

If you deny it, let another line AH be parallel to BF, and let FH be drawn. Then is the triangle CFH (a) = BCA (b) = CFD. (c) Which is absurd.

## PROP. XLI. Fig. 38.

If a Pgr. AEBC have the same base BC with the triangle BCE, and be between the same parallels DE, BC, then is the Pgr. AEBC double to the triangle BCE. Let

Let the line AC be drawn. Then is the triangle BCA  $\equiv$  BCE; therefore is the Pgr. ABCD  $(b) \equiv 2BCA \equiv 2BCE$ . Which was to be demonstrated.

a 37. 1.  
b 34. 1.  
c 6. ax.

*Schol.*

From hence may the area of any triangle BCE be found, for whereas the area of the Pgr. ABCD is produced by the altitude drawn into the base, therefore shall the area of a triangle be produced by half the altitude drawn into the base, or half the base drawn into the altitude; thus, if the base BC be 8, and the altitude 7, then is the area of the triangle BCE 28.

PROP. XLII. Plate I. Fig. 41.

To make a Pgr. ECGF equal to a triangle given ABC in an angle equal to a right-lined angle given.

Through A  $(a)$  draw AG parallel to BC,  $(b)$  make the angle BCG  $\equiv$  the angle given;  $(c)$  bisect the base BC in E, and draw EF parallel to CG; then is the problem resolved.

a 31. 1.  
b 23. 1.  
c 10. 1.

For (AE being drawn) the angle ECG is equal to the given angle by construction, and the triangle BAC  $(d) \equiv 2AEC (e) \equiv$  Pgr. ECGF. Which was to be done.

d 38. 1.  
e 41. 1.

PROP. XLIII. Fig. 9.

In every Pgr. AGEL, the complements LD, GD, of those Pgrs. CK, KF, which stand about the diameter, are equal one to the other.

For the triangle AEL  $(a) \equiv AEG$ , and the triangle ADC  $(a) \equiv ADB$ ; and the triangle DEK  $(a) \equiv DEF (b)$ . Therefore the Pgr. LD  $\equiv$  DG. Which was to be demonstrated.

a 34. 1.  
b 3. ax.

PROP. XLIV. Fig. 42.

To a given right-line A, to apply a parallelogram FL, equal to a given triangle LBC, in a given angle C.

$(a)$  Make a Pgr. FD equal to the triangle LBC, so that the angle GFE may be equal to C. Produce GF till FH be equal to the line given A. Through H  $(b)$  draw IL parallel to EF, which let DE produced meet in I, let DG produced meet with a right-line drawn from I through F in the point K; thro' K  $(b)$  draw KL parallel to GH, which let EF continued meet at M, and IH at L. Then shall FL be the Pgr. required.

a 42. 1.  
b 31. 1.

B 4

For



c 43. 1.  
d 15. 1.

For the Pgr.  $FL(c) = FD = LBC$ ; (d) and the angle  $MFH = GPE = C$ . Which was to be done.

P R O P. XLV. Plate I. Fig. 43, and 44.

Upon a right-line given  $FG$ , and in a given angle  $A$ , to make a Pgr.  $FL$ , equal to a right-lined figure given  $ABCD$ .

a 44. 1.

b 19. ax.

c constr.

Resolve the right-lined figure given into two triangles  $BAD$ ,  $BCD$ ; then (a) make a Pgr.  $FH = BAD$ , so that the angle  $F$  may be equal to  $A$ .  $FI$  being produced, (a) make on  $HI$  the Pgr.  $IL = BCD$ . Then is the Pgr.  $FL(b) = FH + IL(c) = ABCD$ . Which was to be done.

Schol.

Hence is easily found the excess, whereby any right-lined figure exceeds a less right-lined figure.

P R O P. XLVI. Fig. 6.

Upon a right-line given  $CD$  to describe a square  $DB$ .

a 11. 1.

b 3. 1.

c constr.

d 28. 1.

e constr.

f 33. 1.

g cor. 29. 1.

h 29. def.

(a) Erect two perpendiculars  $CB$ ,  $DA$ ; (b) equal to the line given  $CD$ ; then join  $BA$ , and the thing required is done.

For, whereas the Angle  $C + D(c) =$  two right-angles (d) therefore are  $AD$ ,  $BC$  parallel. But they are also (e) equal; (f) therefore  $AB$ ,  $DC$  are both parallel and equal; therefore the figure  $AC$  is a Pgr. and equilateral. Moreover the angles are all right, (g) because one  $A$ , is right; (b) therefore  $DB$  is a square. Which was to be done.

After the same manner you may easily describe a rectangle contained under two right-lines given.

P R O P. XLVII. Fig. 45.

In right-angled triangles  $BAC$ , the square  $BE$ , which is made on the side  $BC$  that subtends the right-angle  $BAC$ , is equal to both the squares  $BG$ ,  $CH$ , which are made on the sides  $AB$ ,  $AC$ , containing the right-angle.

a 12. ax.

b 29. def.

c 4. 1.

d 41. 1.

Join  $AE$ , and  $AD$ ; and draw  $AM$  parallel to  $CE$ .

Because the angle  $DBC(a) = FBA$ , add the angle  $ABC$  common to them both; then is the angle  $ABD = FBC$ . Moreover,  $AB(b) = FB$ , and  $BD(b) = BC$ ; (c) therefore is the triangle  $ABD = FBC$ . But the Pgr.  $BM(d) =$   
 $\frac{1}{2} ABD,$

ABD, and the Pgr. (d)  $BG = 2FBC$  (for GAC is one right-line, by Hypothesis, and 14. 1.) (e) therefore is the Pgr.  $BM = BG$ . By the same way of argument is the Pgr.  $CM = CH$ . Therefore is the whole  $BE = (f) BG + CH$ . Which was to be demonstrated.

d 41. 1.

e 6. ax.

f 2. ax.

Schol.

This most excellent and useful theorem hath deserved the title of *Pythagoras* his theorem, because he was the inventor of it. By the help of which the addition and subtraction of squares are performed; to which purpose serve the two following problems.

PROBLEM I. Plate I. Fig. 46.

To make one square equal to any number of squares given.

Let three squares be given, whereof the sides are AB, BC, CE. (a) Make the right-angle FBZ, having the sides infinite; and on them transfer AB and BC; join AC, then is  $ACq (b) = ABq + BCq$ . Then transfer AC from B to X, and CE the third side given from B to E; join EX. (b) Then is  $EXq = EBq (CEq) + BXq (ACq) (c) = CEq + ABq + BCq$ . Which was to be demonstrated.

Andr.

Tacq.

a 11. 1.

b 47. 1.

c 2. ax.

PROBLEM II. Fig. 2.

Two unequal right-lines being given AE, EG, to make a square equal to the difference of the two squares of the given lines AE, EG.

From the center E, at the distance of AE, describe a circle, and from the point G erect a perpendicular FG, meeting the circumference in F; and draw EF. (a) Then is  $EFq (EAq) = EGq + FGq$ ; (b) Therefore  $EAq - EGq = GFq$ . Which was to be done.

a 47. 1.

b 3. ax.

PROBLEM III. Fig. 45.

Any two sides of a right-angled triangle ABC, being known, to find out the third.

Let the sides AB, AC, encompassing the right-angle, be, the one 6 foot, the other 8. Therefore, whereas  $ACq + ABq = 64 + 36 = 100 = BCq$ , thence is  $BC = \sqrt{100} = 10$ .

47. 1.

Now



Now, let the sides AB, BC, be known, the one 6 foot, the other 10. Therefore since  $BCq = ABq = 100 - 36 = 64 = ACq$ , whence  $AC = \sqrt{64} = 8$ . Which was to be done.

47. 1.

## P R O P. XLVIII. Plate I. Fig. 47.

If the square made upon one side BC of a triangle be equal to the squares made on the other sides of the triangle AB, AC, then the angle BAC comprehended under the two other sides of the triangle AB, AC, is a right-angle.

Perpendicular to AC draw  $AD = AB$ , and join CD.

a 47. 1.

\* See the following theor.

b 8. 1.

c constr.

Now is (a)  $CDq = ADq + ACq = ABq + ACq = BCq$ . \* Therefore is  $CD = BC$ . And therefore the triangles CAB, CAD, are mutually equilateral. Wherefore the angle CAB (b) = CAD (c) = a right-angle. Which was to be demonstrated.

Schol.

We assumed in the demonstration of the last Proposition,  $CD = BC$ , because  $CDq$  was equal to  $BCq$ : Our assumption we prove by the following

## T H E O R E M. Fig. 48.

The squares AF, CG of equal right-lines AB, CD, are equal one to the other: And the sides AB, CD, of equal squares AF, CG, are equal one to the other.

a 34. 1.

b 4. 1. &amp;

6. ax.

a 46. 1.

b 1. part,

c hyp.

d 9. ax.

1. Hypothesis. Draw the diameters EB, HD. Then it is evident that AF is (a) equal to the triangle EAB twice taken, and (b) equal to the triangle HCD twice taken, and equal to (a) CG. Which was to be demonstrated.

2. Hyp. If it may be, let CD be greater than AB. Make  $CT = AB$ , and let  $CS = CTq$ . Therefore is  $CS$  (b) = EB (c) = CG. Which is absurd.

Corol.

After the same manner any rectangles equilateral one to another, are demonstrated to be also equal.

The End of the first Book.

The

## The SECOND BOOK

O F

EUCLIDE'S  
ELEMENTS.

## Definitions.

I. **E**very right-angled Parallelogram ABCD, is said to be contained under two right-lines AB, AC, comprehending a right-angle.

Plate I.  
Fig. 7.

Therefore when you meet with such as these, the rectangle under BA, AC, or more briefly the rectangle BAC, or  $BA \times AC$  (or ZA, for  $Z \times A$ ) the rectangle meant is that which is contained under the right-lines BA, AC, set at right-angles.

II. In every Pgr. ALGE, any one of those parallelograms which are about the diameter, together with its two complements is called a Gnomon. As the Pgr.  $FB + DL + FK$  (BEC) is a Gnomon; and likewise the Pgr.  $FB, DL + BC$  (FAK) is a Gnomon.

Fig. 9.

## P R O P. I. Fig. 49.

If two right-lines AF, AB, are given, and one of them AB be divided into as many parts or segments as you please; the rectangle comprehended under the two whole right-lines AB, AF, shall be equal to all the rectangles contained under the whole line AF, and the several segments, AD, DE, EB.

(a) Set AF perpendicular to AB. Thro' F (a) draw an infinite line FG perpendicular to AF. From the points D, E, B, erect perpendiculars DH, EI, BG. Then is AG a rectangle comprehended under AF, AB, and is (b) equal to the rectangles AH, DI, EG, that is (because DH, EI, AF, (c) are equal) to the rectangles under AF, AD, under AF, DE, under AF, EB. Which was to be demonstrated,

a 11. 1.

b 19 ax. 1.  
c 34. 1.

Schol.



If two right-lines given are both divided into how many parts soever, one whole multiplied into the other shall bring out the same product, as the parts of one multiplied into the parts of the other.

a 1. 2.  
b 2. ax.

For let  $Z$  be  $= A + B + C$ , and  $Y = D + E$ ; then, because  $DZ$  (a)  $= DA + DB + DC$ , and  $EZ$  (a)  $= EA + EB + EC$ , and  $YZ$  (a)  $= DZ + EZ$ , (b) Therefore  $ZY$  will be  $= DA + DB + DC + EA + EB + EC$ . Which was to be demonstrated.

From hence we have a method of multiplying compound lines into compound ones. For if the rectangles of all the parts be taken, their sum shall be equal to the rectangle of the wholes.

1. 2.

But whensoever in the multiplication of lines into themselves you meet with these signs—intermingled with these  $+$ , you must also have particular regard to the signs. For of  $+$  multiplied into — ariseth —; but of — into — ariseth  $+$  ex. gr. let  $+$   $A$  be multiplied into  $-C$ ; then because  $+$   $A$  is not affirmed of all  $B$ , but only of that part of it, whereby it exceeds  $C$ , therefore  $AC$  must remain denied; so that the product will be  $AB - AC$ . Or thus; because  $B$  consists of the parts  $C$  and  $B - C$ , \* thence  $AB = AC + A \times B - C$ , take away  $AC$  from both, then  $AB - AC = A \times B - C$ . In like manner, if —  $A$  be to be multiplied into  $B - C$ , then since by virtue of the sign —,  $A$  is not denied of all  $B$ , but only of so much as it exceeds  $C$ , therefore  $AC$  must remain affirmative, whence the product will be  $-AB + AC$ . Or thus; because  $AB = AC + A \times B - C$ ; take away all from both sides, and there will be  $-AB = -AC - A \times B - C$ ; add  $AC$  to both, and it will be  $-AB + AC = -A \times B - C$ .

This being sufficiently understood, the nine following propositions, and innumerable others of that kind arising from the comparing of lines multiplied into themselves (which you may find done to your hand in *Vieta*, and other analytical Writers) are demonstrated with great facility, by reducing the matter for the most part to almost a simple work.

19. ax.

Furthermore, \* it appears that the product arising from the multiplication of any magnitude into the parts of any number is equal to the product arising from the multiplication of the same into the whole number.

As  $5A + 7A = 12A$ , and  $4A \times 5A + 4A \times 7A =$   
4 A

A X 12 A. Wherefore what is here delivered of the multiplying of right-lines into themselves, the same may be understood of the multiplying of numbers into themselves, so that whatsoever is affirmed concerning lines in the nine following Theorems, holds good also in numbers; seeing they all immediately depend on, and are deriv'd from this first.

PROP. II. Plate I. Fig. 50.

If a right-line Z be divided any wise into two parts, the rectangles comprehended under the whole line Z, and each of the segments A, E, are equal to the square made of the whole line Z.

I say that  $ZA + ZE = Zq$ . For take  $B = Z$ ; (a) then is  $BA + BE = BZ$ , that is (because  $B = Z$ )  $ZA + ZE = Zq$ . Which was to be demonstrated.

a 1. 2.

PROP. III. Fig. 50.

If a right-line Z, be divided any wise into two parts, the rectangle comprehended under the whole line Z, and one of the segments E, is equal to the rectangle made of the segments A, E, and the square described on the said segment E.

a 1. 2.

I say  $ZE = AE + Eq$ . (a) For  $EZ = EA + Eq$ .

PROP. IV. Fig. 50.

If a right-line Z be cut any wise into two parts, the square described on the whole line Z, is equal to the squares described on the segments A, E, and to twice a rectangle made of the segments A, E, taken together.

I say that  $Zq = Aq + Eq + 2AE$ . For  $ZA(a) = Aq + AE$ , and  $ZE(a) = Eq + EA$ . Therefore whereas  $ZA + ZE(b) = Zq$ , (c) thence is  $Zq = Aq + Eq + 2AE$ . Which was to be demonstrated.

a 3. 2.

b. 2. 2.

c 1. ax.

Otherwise thus; Upon the right-line AB make the square AD, and draw the diameter EB; thro' C, the point wherein the line AB is divided, draw the perpendicular CF; and thro' the point G draw HI parallel to AB.

Fig. 51.

Because the angle  $EHG = A$  is a right-angle, and  $AEB$  is (d) half a right (e) therefore is the remaining angle  $HGE$  half a right-angle. Therefore is  $HE(f) = HG(g) = EF(g) = AC$ , so that  $HF(b)$  is the square of the right-line AC. After the same manner is CE proved to

d 4. cor.

32. 1.

e 32. 1.

f 6. 1.

g 34. 1.

h 29. def.



to be  $CBq$ . Therefore  $AG$ ,  $GD$ , are rectangles under  
 k 19. ax. 1.  $AC$ ,  $CB$ , wherefore the whole square  $AD$  ( $k$ ) =  $ACq$   
 $+ CBq + 2ACB$ . Which was to be demonstrated.

Coroll.

1. Hence it appears that the Pgrs. which are about  
 the diameter of a square are also squares themselves.

2. That the diameter of any square bisects it angles.

3. That if  $a = \frac{1}{2} Z$ , then is  $Zq = 4 Aq$ , and  $Aq =$   
 $\frac{1}{4} Zq$ . As on the contrary, if  $Zq = 4 Aq$ , then is  $A =$   
 $\frac{1}{2} Z$ .

P R O P. V.

A ——— | ——— | ——— B. If a right-line  $AB$  be cut in-  
                   C        D                    to equal parts  $AC$ ,  $CB$ , and in-  
 to unequal parts  $AD$ ,  $DB$ , the rectangle comprehending under  
 the unequal parts  $AD$ ,  $DB$ , together with the square that is  
 made of the difference of the parts  $CD$ , is equal to the  
 square described on the half line  $CB$ .

I say that  $CBq = ADB + CDq$ .

a 4. 1.  
 b 3. 2.  
 c hyp.  
 d 1. 2.

For these are  $\left\{ \begin{array}{l} (a) CDq + CDB + DBq + CDB. \\ (b) CDq + (c) (AC \times BD) + CDB. \\ (d) CDq + (d) ADB. \end{array} \right.$   
 all equal.

This theorem is somewhat differently express'd and  
 more easily demonstrated thus; A rectangle made of the  
 sum and the difference of two right-lines  $A$ ,  $E$ , is equal to  
 the difference of the squares of those lines.

\* Sch. 1. 2. For if  $A + E$  be multiplied into  $A - E$ , \* there ariseth  
 $Aq - AE + EA - Eq = Aq - Eq$ . Which was to be de-  
 monstrated.

Schol. Plate I. Fig. 52.

If the line  $AB$  be divided otherwise, (*viz.*) nearer to  
 the point of bisection, in  $E$ ; then is  $AEB \sqsubset ADB$ .

a 5. 2. &  
 3. ax.

For  $AEB (a) = CBq - CEq$ , and  $ADB (a) = CBq -$   
 $CDq$ . Therefore, whereas  $CDq \sqsubset CEq$ , thence is  $AEB$   
 $\sqsubset ADB$ . Which was to be demonstrated.

Coroll.

b 4. 2.

1. Hence is  $ADq + DBq \sqsubset AEq + EBq$ . For  $ADq$   
 $+ DBq + 2ADB (b) = ABq (b) = AEq + EBq + 2AEB$ .  
 Therefore because  $2AEB \sqsubset 2ADB$ , shall  $ADq + DBq$   
 $\sqsubset AEq + EBq$ . Which was to be demonstrated.

c 3. ax.

2. Hence is  $ADq + DBq - AEq (c) - EBq = 2AEB -$   
 $2ADB$ .

P R O P.

PROP. VI. Plate I. Fig. 53.

If a right-line A be divided into two equal parts, and another right-line E, added to the same directly in one right-line, then the rectangle comprehended under the whole and the line added, (viz.  $A+E$ ), and the line added E, together with the square which is made of  $\frac{1}{2}$  the line A, is equal to the square of  $\frac{1}{2} A+E$  taken as one line.

I say that  $\frac{1}{4} Aq. (a) Q. \frac{1}{2} A + AE + Eq = Q. \frac{1}{2} A + E. (a) For, Q. \frac{1}{2} A + E = \frac{1}{4} Aq + Eq + AE. Which$  was to be demonstrated.

a 4. 3.

Cor. 4. 2.

Coroll.

Hence it follows, that if 3 right-lines E,  $E+\frac{1}{2}A$ ,  $E+A$  be in arithmetical proportion, then the rectangle contained under the extreme terms E,  $E+A$ , together with the square of the difference  $\frac{1}{2}A$ , is equal to the square of the middle term  $E+\frac{1}{2}A$ .

PROP. VII. Fig. 50.

If a right-line Z be divided any wise into two parts, the square of the whole line Z, together with square made of one of the segments E, is equal to a double rectangle comprehended under the whole line Z, and the said segment E, together with the square made of the other segment A.

I say, that  $Zq + Eq = 2ZE + Aq. For Zq (a) = Aq + Eq + 2AE, and 2ZE (b) = 2Eq + 2AE. Which$  was to be demonstrated.

a 4. 2.

b 3. 2.

Coroll.

Hence it follows, that the square of the difference of any two lines Z, E, is equal to the square of both the lines less by a double rectangle comprehended under the said lines.

(c) For  $Zq + Eq - 2ZE = Aq = Q. Z - E.$

c 7. 2. and  
3. ax.

PROP. VIII. Fig. 50.

If a right-line Z be divided any wise into two parts, the rectangle comprehended under the whole line Z, and one of the segments E four times, together with the square of the other segment A, is equal to the square of the whole line Z, and the segment E, taken as one line  $Z+E$ .

I say, that  $4ZE + Aq = Q. Z + E. For 2ZE (a) = Zq + Eq - Aq. Therefore 4ZE + Aq = Zq + Eq + 2ZE (b) = Q. Z + E. Which$  was to be demonstrated.

a 7. 2. and  
3. ax.  
b 4. 2.

PROP.



## PROP. IX. Plate I. Fig. 54.

If a right-line AB be divided into equal parts AC, CB and into unequal parts AD, DB, then are the squares of the unequal parts AD, DB, together, double to the square of the half line AC, and to the square of the difference CD.

a 4. 2.  
b hyp.  
c 7. 2.  
d 2. ax.

I say that  $AD^2 + DB^2 = 2 AC^2 + 2 CD^2$ . For  $AD^2 + DB^2 (a) = AC^2 + CD^2 + 2 ACD + DB^2$ . But  $2 ACD (b) (2 BCD) + DB^2 (c) = CB^2 (AC^2) + CD^2$ . (d) Therefore  $AD^2 + DB^2 = 2 AC^2 + 2 CD^2$ . Which was to be demonstrated.

This may be otherwise delivered and more easily demonstrated thus; the aggregate of the squares made of the sum and the difference of two right-lines A, E, is equal to the double of the squares made from those lines.

a 4. 2.  
b cor. 7. 2.

For  $Q. A + E (a) = Aq + Eq + 2 AE$ , and  $Q. A - E (b) = Aq + Eq - 2 AE$ . These added together make  $2 Aq + 2 Eq$ . Which was to be demonstrated.

## PROP. X. Fig. 53.

If a right-line A be divided into two equal parts, and another line be added in a right-line with the same, then is the square of the whole line together with the added line (as being one line) together with the square of the added line E, double to the square of half A, and the added line E, taken as one line, ~~plus half A square~~.

a 4. 2.  
b cor. 4. 2.  
c 4. 2.

I say that  $Eq + Q. A + E, i. e. (a) Aq + 2 Eq + 2 AE = 2 Q. \frac{1}{2} A + 2 Q. \frac{1}{2} A + E$ , [For  $2 Q. \frac{1}{2} A (b) = \frac{1}{2} Aq$ . And  $2 Q. \frac{1}{2} A + E (c) = \frac{1}{2} Aq + 2 Eq + 2 AE$ . Which was to be demonstrated.

## PROP. XI. Fig. 55.

To cut a right-line given AB, in a point G, so that the rectangle comprehended under the whole line AB, and one of the segments BG, shall be equal to the square, that is made of the other segment AG.

a 46. 1.  
b 10. 1.

Upon AB (a) describe the square AC (b) Bisect the side AD in E, and draw the line EB; from the line EA produced take  $EF = EB$ . On AF make the square AH. Then is  $AH = AB \times BG$ .

For HG being drawn out to I; the rectangle DH + Aq (c) = EFq (d) = EBq (e) = BAq + EAq: Therefore DH (f) = BAq = to the square AC. Take away AI common to both, there remains the square AH = GC, that is, AGq = AB x BG. Which was to be done.

c 6. 2.  
d constr.  
e 47. 1.  
f 3. ax.

Schol.

This proposition cannot be performed by numbers; for there is no number that can be so divided, that the product of the whole into one part shall be equal to the square of the other part.

\* 6. 13.

PROP. XII. Plate I. Fig. 56.

In obtuse-angled triangles as ABC, the square that is made of the side AC, subtending the obtuse angle ABC, is greater than the squares of the sides BC, AB, that contain the obtuse angle ABC; by a double rectangle contained under one of the sides BC, which are about the obtuse angle ABC, on which side produced the perpendicular AD falls, and under the segment BD, taken without the triangle from the point on which the perpendicular AD falls to the obtuse angle ABC.

I say that ACq = CBq + ABq + 2CB x BD.

For these are all equal { ACq.  
(a) CDq + ADq.  
(b) CBq + 2CBD + BDq + ADq.  
CBq + 2CBD + (c) ABq.

a 47. 1.  
b 4. 2.  
c 47. 1.

Scholium.

Hence, the sides of any obtuse-angled triangle ABC being known, the segment BD intercepted betwixt the perpendicular AD, and the obtuse angle ABC, as also the perpendicular itself AD, shall be easily found out.

Thus, Let AC be 10, AB 7, CB 5. Then is ACq 100, Bq 49, CBq 25. And ABq + CBq = 74. Take that out of 100, then will 26 remain for 2 CBD. Wherefore CBD shall be 13; divide this by CB 5, there will be found for BD. Whence AD will be found out by the 47. 1.

PROP. XIII. Fig. 57.

In acute-angled triangles as ABC, the square made of the side AB, subtending the acute angle ACB, is less than the squares made



made of the sides AC, CB, comprehending the acute angle ACB, by a double rectangle contained under one of the sides BC, which are about the acute angle ACB, on which the perpendicular AD falls, and under the line DC, taken within the triangle from the perpendicular AD, to the acute angle ACB.

I say that  $ACq + BCq = ABq + 2BCD$ .

a 47. 1.

b 7. 2.

c 47. 1.

For these are equal.

$$\begin{cases} ACq + BCq. \\ (a) ADq + DCq + BCq. \\ (b) ADq + BDq + 2BCD. \\ (c) ABq + 2BCD. \end{cases}$$

Coroll.

Hence, the sides of an acute-angled triangle ABC being known, you may find out the segment DC, intercepted betwixt the perpendicular AD, and the acute angle ACB, as also the perpendicular it self AD.

Let AB be 13, AC 15, BC 14. Take ABq (169) from  $ACq + BCq$ , that is, from  $225 + 196 = 421$ . Then remains 252 for  $2BCD$ , wherefore BCD will be 126, divide this by BC 14, then will 9 be found out for DC. From whence it follows,  $AD = \sqrt{225 - 81} = 12$ .

PROP. XIV. Plate I. Fig. 58.

To find a square MC equal to a right-lined figure given A.

(a) Make the rectangle DB=A, and produce the greater side thereof DC to F, so that CF=CB, (b) bisect DF, in G, about which as the center, at the distance of GF, describe the circle FHD, and draw out BC, till it meets the circumference in H. Then shall be  $CHq = MC = A$ .

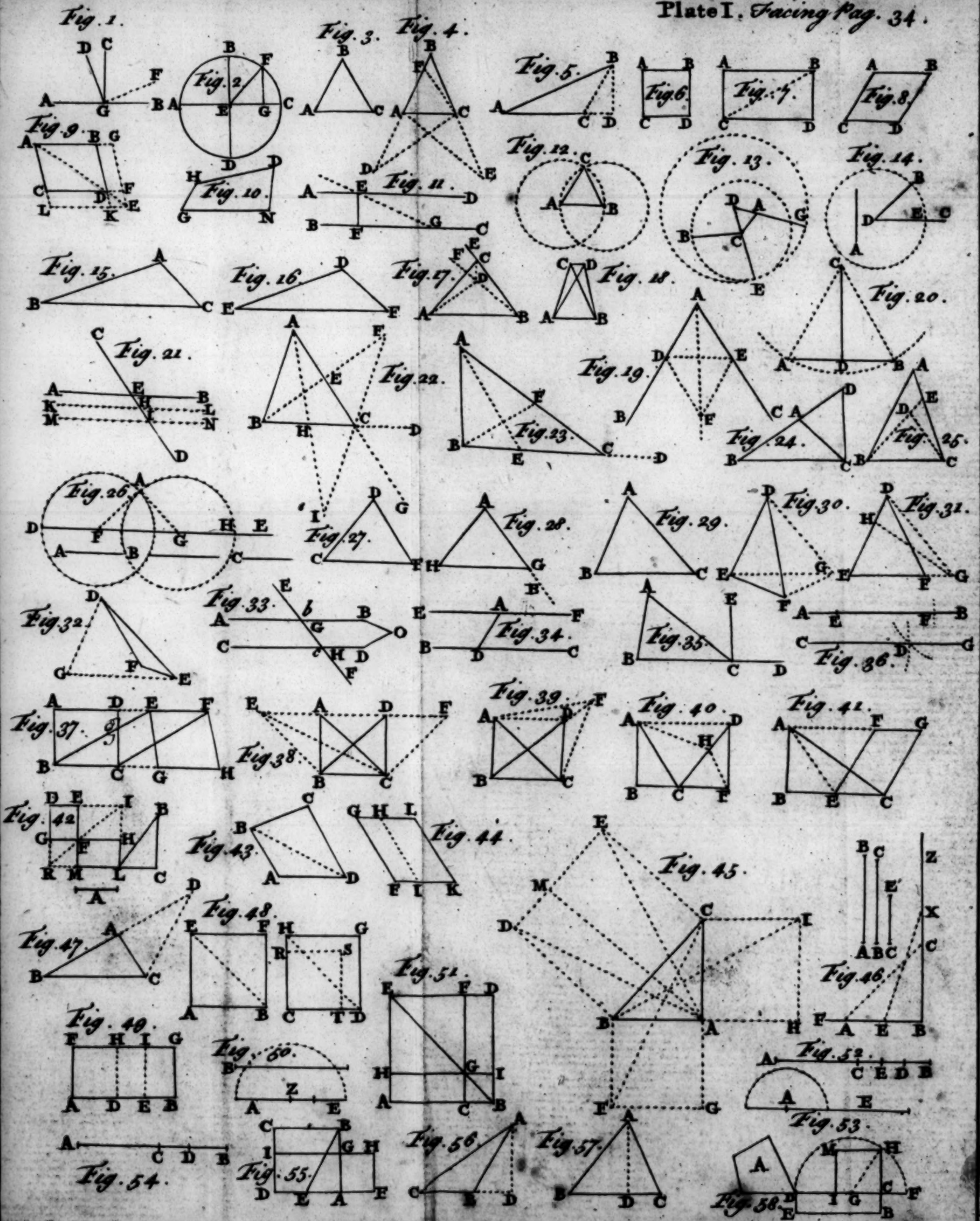
For let GH be drawn. Then is  $A (c) = DB (c) = DCF (d) = GFq - GCq (e) = HCq (c) = MC$ . Which was to be done.

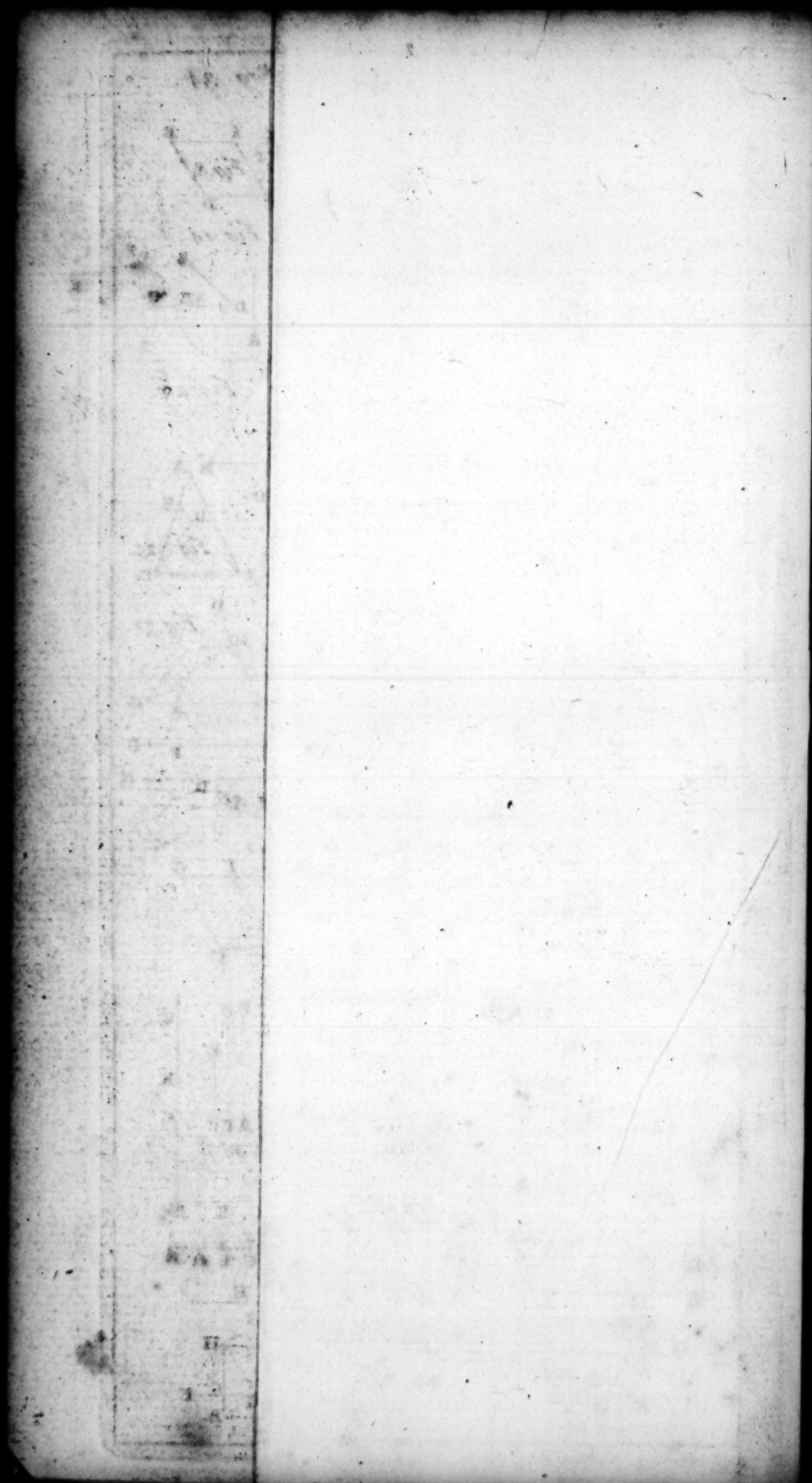
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# The THIRD BOOK OF EUCLIDE'S ELEMENTS.

## Definitions.

I. **E**Qual circles (GABC, HDEF) are such whose diameters are equal; or, from whose centers right-lines drawn GA, HD, are equal.

Plate II.  
Fig. 1, 2.

II. A right-line MO, is said to touch a circle GABC, when touching the same, and being produced, it cutteth it not.

Fig. 2.

The right-line GCO cuts the circle GABC.

III. Circles are said to touch one the other, which touch, but cut not one the other. As Fig. 8.

Those of Fig. 7. cut each other.

IV. In a circle GABD, right-lines FE, KL, are said to be equally distant from the center, when perpendiculars GH, GN, drawn from the center G to them, are equal. And that line BC is said to be furthest distant from it, on which the greater perpendicular GI falls.

Fig. 3.

V. A segment of a circle (BPC) is a figure contained under a right-line BC, and a portion of the circumference of a circle BPC.

Fig. 2.

VI. An angle of a segment CBP, is that angle which is contained under a right-line BC, and the circumference of a Circle PB.

VII. An angle BPC is said to be in a segment BPC, when in the circumference thereof some point P is taken, and from it right-lines BP, CP, drawn to the ends of the right-line BC, which is the base of the segment; then the angle BPC contained under the adjoined lines BP, CP, is said to be an angle in a segment.

VIII. But when the right-lines BP, CP, comprehending the angle BPC, receive any periphery of the circle BAC, then the angle BPC is said to stand upon that periphery.



Fig. 1.

IX. A sector of a circle (HRS) is when an angle RHS is set at the center H of that circle; namely, that figure RHS comprehended under the right-lines RH, SH, containing the angle, and the part of the circumference RS, intercepted between them.

Fig. 1, 2,

X. Like segments of a circle (ABC, DFE) are those which include equal angles (ABC, DEF); or, in which the angles ABC, DEF, are equal.

## PROP. I. Plate II. Fig. 4.

To find the center F of a given circle ABC.

Draw a right-line AC any-wise in the circle, which bisect in E; thro' E draw a perpendicular DB, and bisect the same in F; the point F shall be the center.

a 15. def. 1.

b 8. 1.

c 10. def. 1.

d 12. ax.

e 9. ax.

If you deny it, let G, a point without the line BD, be the center (for it cannot be in the line BD, since that is divided unequally in every point but F) let the lines GA, GC, GE, be drawn. Now if G be the center, (a) then is  $GA = GC$ , and  $AE = EC$ , by construction, and the side GE common. (b) Therefore are the angles GEA, GEC, equal, and (c) consequently right. (d) Therefore the angle  $GEC = FEG$ . (e) Which is absurd.

## Coroll.

Hence, if in a circle a right-line BD bisect any right-line AC at right-angles, the center shall be in the cutting line BD.

Fig. 5.

Andr.

Tacq.

The center of a circle is easily found out by applying the top of a square to the circumference thereof. For if the right-line GH that joins the points G, H, in which the sides of the square GB, BH, cut the circumference, be bisected in E, the point E shall be the center. The demonstration whereof depends upon Prop. XXXI. of this Book.

## PROP. II. Fig. 2.

If in the circumference of a circle GAC, any two points A, C, be taken, the right-line AC, which joins those two points, shall fall within the circle.

a 15. def. 1.

b 5. 2.

c 16. 2.

d 19. 1.

Take in the right-line AC any point S; from the center G draw GA, GS, GC. Because  $GA = GC$ , therefore is the angle  $A = C$ . But the angle  $GSC = \angle A$ , therefore is  $GSC = C$ , therefore  $GC = GS$ . But

But GC only reaches the circumference, therefore GS comes not so far; wherefore the point S is within the circle. The same may be proved of any other point in the line AC. And therefore the whole line AC falls within the circle. *Which was to be dem.*

*Coroll.*

Hence, if a right-line touch a circle, without cutting it, it touches but in one point.

PROP. III. Plate II. Fig. 5.

*If in a circle EABC, a right-line BD drawn thro' the center, bisects any other line AC, not drawn thro' the center, it shall also cut it at right-angles: And if cuts it at right-angles, it shall also bisect the same.*

From the center E let the lines EA, EC, be drawn.

1. *Hyp.* Because AF (a) = FC, and EA (b) = EC, and the side EF common; the angles EFA, EFC, (c) shall be equal, and (d) consequently right. *Which was to be demonstrated.*

2. *Hyp.* Because EFA (e) = EFC, and the angle EAF (f) = ECF, and the side EF common; (g) therefore is AF = FC. Therefore AC is cut into two equal parts. *Which was to be demonstrated.*

*Coroll.*

Hence, in any equilateral or Isosceles triangle, if a line drawn from the vertical angle bisect the base, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the vertical angle bisects the base.

PROP. IV. Fig. 6.

*If in a circle ACD, two right-lines AB, CD, cut each other, and neither of them pass thro' the center E, they shall not cut each other into equal parts.*

For if one line pass thro' the center, 'tis plain it cannot be bisected by the other; because by hypothesis, the other does not pass thro' the center.

If neither of them pass thro' the center, then from the center E draw EF; now if AB, CD, were both bisected in F, then (a) would the angles EFB, EFD, be both right, and consequently equal. (b) *Which is absurd.*

C 3

PROP.

a hyp.  
b 15. def. 1.  
c 8. 1.  
d 10. def. 1.  
e hyp. and  
12. ax.  
f 5. 1.  
g 26. 1.

a 3. 3.  
b 9. ax.



## PROP. V. Plate II. Fig. 7.

If two circles BAC, BDC, cut each other, they shall not have the same center E.

a 15. def. 1.  
b 9. ax.

For otherwise the lines EB, EDA, drawn from E, the common center, would be  $DE (a) = EB (a) = EA (b)$  Which is absurd.

## PROP. VI. Fig. 8.

If two circles BAC, BDE, inwardly touch each other (in B) they have not one and the same center F.

a 15. def. 1.  
b 9. ax.

For otherwise the right-lines FB, FDA, drawn from the center F, would be  $FD (a) = FB (a) = FA (b)$  Which is absurd.

## PROP. VII. Fig. 9.

If in AB, the diameter of a circle, some point G be taken, which is not the center of the circle, and from that point certain right-lines GC, GD, GE, fall on the circle, the greatest line shall be that (GA) which passeth thro' the center F; the least, the remainder of the same line (GB.) And of all the other lines, the line GC, nearest to that which was drawn thro' the center is always greater than any line farther removed GD; and there can but two equal lines fall from the same point on the circle, viz. one on each side of the least GB, or of the greatest GA.

\* 23. 1.  
a 20. 1.

From the center F draw the right-lines FC, FD, FE; make the angle  $BFH = BFE$ .

b 15. def.  
c 9. ax.  
d 24. 1.  
e 20. 1.  
f 5. ax.

1.  $GF + FC$  (that is GA)  $(a) \sqsubset GC$ . Which was to be demonstrated.

2. The side FG is common, and  $FC (b) = FD$ , and the angle  $GFC (c) \sqsubset GFD$ ;  $(d)$  wherefore the base  $GC \sqsubset GD$ .

3.  $FB (FE) (e) \sqsupset GB + GF$ . Therefore FG, which is common, being taken away from both, there remains  $BG \sqsupset EG$ .

g const.  
h 4. 1.

4. The side FG is common, and  $FE = FH$ , and the angle  $BFH (g) = BFE$ ;  $(h)$  Therefore is  $GE = GH$ . But that no other line GD from the point G, can be equal to GE, or GH, is already proved. Which was to be demonstrated.

PROP.

PROP. VIII. Plate II. Fig. 10.

If some point A be taken without a circle, and from that point be drawn certain right-lines AI, AH, AG, AF, to the circle, and of those one AI, be drawn thro' the center K, and the others any wise; of all those lines that fall on the concave of the circumference, that is the greatest (AI) which is drawn thro' the center; and of the others, that (AH) which is nearest to the line that passes thro' the center, is greater than that which is more distant, AG. But of all those lines that fall on the convex part of the circle, the least is that (AB) which is drawn from the point A, to the diameter IB; and of the others, that (AC) which is nearest to the least, is less than that which is farther distant AD. And from that point there can be only two equal right-lines AC, AL, drawn, which shall fall on the circumference on each side of the least line AB, or of the greatest AI.

From the center K, draw the right-lines KH, KG, KF, KC, KD, KE, and make the angle  $AKL = AKC$ .

1.  $AI (AK + KH) (a) \subset AH$ . a 20. 1.
2. The side AK is common, and  $KH = KG$ , and the angle  $AKH \subset AKG$ ; (b) therefore the base  $AH \subset AG$ . b 24. 1.
3.  $KA (c) \supset KC + CA$ . From hence take away KC, KB, which are equal; then will remain AB, (d)  $\supset AC$ . c 20. 1.  
d 5. ax.
4.  $AC + CK (e) \supset AD + DK$ . Take from both, CK, DK, which are equal; then remains  $AC (f) \supset AD$ . e 21. 1.  
f 5. ax.
5. The side KA is common, and  $KL = KC$ , and the angle  $AKL (g) = AKC$ ; (b) therefore  $LA = CA$ . But that no other line could be drawn equal to these, was proved above. Therefore, &c. g constr.  
h 4. 1.

PROP. IX. Fig. 11.

If in a circle BCK, a point A be taken, and from that point more than two equal right-lines AB, AC, AK, can be drawn to the circumference, then is that point A the center of the circle.

For (a) from no point without the center can more than two equal right-lines be drawn to the circumference. Therefore A is the center. Which was to be demonstrated. a 7. 3.

## PROP. X. Plate II. Fig. 12.

A circle IAKBL, cannot cut another circle IEKFL, in more than two points.

a cor. I. 3.

Let one circle, if it may be, cut the other in three points I, K, L, and IK, KL, being join'd, let them be bisected in M and N. (a) Both circles have their centers in their perpendiculars MC, NH, and in the intersection of those perpendiculars which is O. Therefore the circles that cut each other have the same center. Which is false, by Prop. 5. 3.

## PROP. XI. Fig. 8.

If two circles FBDE, GABC, touch one the other inwardly, and their centers be taken G, F; a right-line FG joining their centers, and produced, shall cut the circumference in B, the point of contact of the circles.

a 15. def. 1.

b 7. 3.

c 9. ax.

If it can be, let the right-line FG produced cut the circles in some other point than B; so that not GFB, but GFIK, shall be a right-line. Let the line FB be drawn. Now, because FB (a) = FI, and FK (b)  $\perp$  GB (since the right-lines GFB passes thro' G, the center of the greater circle) therefore is FI  $\perp$  FK. (c) Which is absurd.

## PROP. XII. Fig. 13.

If two circles ACD, BCE, touch one the other outwardly, the right-line AB, which joins their centers A, B, shall pass thro' the point of contact C.

a 30. 1.

b 9. ax.

If it may be, let ADEB be a right-line cutting the circles, not in the point of contact C, but in the points D, E; draw AC, CB, then is AD  $\perp$  EB (AC  $\perp$  CB) (a)  $\perp$  ADEB. (b) Which is absurd.

## PROP. XIII. Fig. 8.

A circle GCB cannot touch a circle FBE in more points than one B, whether it be inwardly or outwardly.

a 11. 3.

b 15. def. 1.

c 15. def. 1.

d 9. ax.

1. Let one circle (if it can be) touch another in two points B, E, (a) Then will the right-lines GF, that joins the centers, if it be produced, fall as well in B as E. Now because GE (b) = GB, and FE  $\perp$  GE, therefore is FB (c) FE)  $\perp$  GB. (d) Which is absurd.

e 2. 3.

2. If it be said to touch outwardly in the points L and M, then draw the line LM, (e) which will be in both circles.



circles. Therefore those circles cut one the other; which is against the Hyp.

PROP. XIV. Plate II. Fig. 14.

In a circle EABC, equal right-lines AC, BD, are equally distant from the center E: And right-lines AC, BD, which are equally distant from the center, are equal among themselves.

From the center E, draw the perpendiculars EF, EG, (a) which will bisect the lines AC, BD; join EA, EB.

1. Hyp.  $AC = BD$ , therefore  $AF (b) = BG$ . But also  $EA = EB$ ; therefore  $FEq (c) = EAq - AFq = EBq - BGq (c) = EGq$ . (d) Therefore  $FE = EG$ .

2. Hyp.  $EF = EG$ . Therefore  $AFq (e) = EAq - EFq = EBq - EGq = BGq$ . Therefore  $AF (d) = GB$ , and (e) consequently  $AC = BD$ . Which was to be demonstrated.

a 3. 3.  
b 7. ax.  
c 47. 1.  
and 3. ax.

d schol.  
48. 1.  
e 6. ax.

PROP. XV. Fig. 15.

In a circle GABC, the greatest line is the diameter AD; and of all other lines, that FE, which is nearest to the center G, is greater than any line BC farther distant from it.

1. Draw GB, and GC. The diameter AD (a)  $GB + GC (b) < BC$ .

2. Let the distance GI be  $< GH$ . Take  $GN = GH$ . Thro' the point N draw KL, perpendicular to GI: Join GK, GL. Because  $GK = GB$ , and  $GL = GC$ , and the angle  $KGL < BGC$ ; (c) therefore is  $KL (FE) < BC$ . Which was to be demonstrated.

a 15. def. 1.  
b 20. 1.

c 24. 1.

PROP. XVI. Fig. 16.

A line CD, drawn from the extreme point of the diameter HA, of a circle BALH, perpendicular to the said diameter, shall fall without the circle; and between the same right-line and the circumference, cannot be drawn another line AL. And the angle of the semicircle BAI, is greater than any right-lined acute angle BAL; and the remaining angle without the circumference DAI, is less than any right-lined angle.

1. From the center B, to any point F, in the right-line AC, draw the right-line BF. The side BF subtending the right-angle BAF, is (a) greater than the side BA, which is opposite to the acute-angle BFA. Therefore, whereas BA, (BG) reaches to the circumference, BF, shall reach further; and so the point F, and for the same

a 19. 1.

same reason any other point of the line AC, shall be without the circle.

b 19. 1.

2. Draw BE perpendicular to AL. The side BA, opposite to the right-angle BEA, is (b) greater than the side BE, which subtends the acute-angle BAE; therefore the point E, and so the whole line EA, falls within the circle.

3. Hence it follows, that any acute-angle, to wit, EAD, is greater than the angle of contact DAI, and that any acute-angle BAL is less than the angle of a semicircle BAI. Which was to be dem.

Coroll.

Hence, a right-line drawn from the extremity of the diameter of a circle, and at right-angles, is a tangent to the said circle.

From this proposition are gathered many paradoxes, and wonderful confectaries, which you may meet with in the interpreters.

PROP. XVII. Plate II. Fig. 17.

From a point given A, to draw a right-line AC, which shall touch a circle given DBC.

From D, the center of the circle given, to the given point A, let the line DA be drawn, cutting the circumference in B, from the center D, describe another circle thro' the point A; and from B, draw a perpendicular to AD, which shall meet with the circle AE in the point E; and draw ED meeting with the circle BC, in the point C. Then a line drawn from A to C, shall touch the circle DBC.

a 15. def. 1.  
b 4. 1.  
c cor. 16 3.

For DB (a) = DC, and DE (a) = DA, and the angle D is common; (b) therefore the angle ACD = EBD and both right. (c) Therefore AC, touches the circle in C. Which was to be dem.

PROP. XVIII. Fig. 16.

If any right-line CA touches a circle BALH, and from the center to the point of contact A, a right-line BA be drawn; that line BA shall be perpendicular to the tangent CA.

a def. 2. 3.  
b cor. 17. 1.  
c 19. 1.  
d 9. ax.

If you deny it, let some other line BF be drawn from the center B, perpendicular to the tangent; and (a) cutting the circle in G. Therefore, whereas the angle BFA is said to be right, (b) thence the angle BAF acute; (c) so that BA (BG) < BF. (d) Which is absurd.

PROP.

PROP. XIX. Plate II. Fig. 16.

If any right-line CD touch a circle, and from the point of contact A, a right-line AH, be erected at right-angles to the tangent, the center of the circle shall be in the line AH so erected.

If you deny it, let the center be without the line AK, in the point K; and from K, to the point of contact, let KA be drawn. Therefore the angle KAD is right, and (a) consequently equal to the angle HAD, which was right by Hypothesis. (b) Which is absurd.

a 12. ax.  
b 9. ax.

PROP. XX. Fig. 18, 19, 20.

In a circle DABC, the angle BDC at the center, is double of the angle BAC at the circumference, when the same arch of the circle BC, is the base of the angle.

Draw the Diameter ADE. The outward angle BDE (a) = DAB + DBA (b) = 2DAB: In like manner the angle BDC = 2DAC. Therefore in the first case (c) the whole angle BDC = 2BAC, and in the third case the remaining angle BDC (d) = 2BAC. Which was to be demonstrated.

a 32. 1.  
b 5. 1.  
c 2 ax.  
d 20. ax.

PROP. XXI. Fig. 21.

In a circle EDAC, the angles DAC, and DBC, which are in the same segment, are equal one to the other.

1. Case. If the segment DABC be greater than a semicircle, from the center E draw ED, EC. Then is twice the angle A (a) = E (a) = 2B. Which was to be demonstrated.

a 20. 3.

2. Case. If the segment, (Fig. 22.) be less than a semicircle, then is the sum of the angles of the triangle ADF equal to the sum of the angles of the triangle BCF; from each let AFD equal to BFC, (b) and ADB (c) = ACB, be taken away, then remains DAC = DBC. Which was to be demonstrated.

b 15. 1.  
c by the 1st case.

PROP. XXII. Fig. 23.

The angles ADC, ABC, of a quadrilateral figure ABCD, described in a circle, which are opposite one to the other, are equal to two right-angles.

Draw AC, BD. The angle ABC + BCA + BAC (a) = 2 right-angles. But BDA (b) = BCA, and BDC (b) = BAC. (c) Therefore ABC + ADC = 2 right-angles. Which was to be demonstrated. Coroll

a 32. 1.  
b 21. 3.  
c 1. ax.



## Corol.

1. Hence, if one side AB of a quadrilateral, described in a circle, be produced, the external angle EBC is equal to the internal angle ADC, which is opposite to that ABC, and adjacent to EBC, as appears by 13. 1. and 3. ax.

2. A circle cannot be described about a Rhombus because its opposite angles are greater; or less than two right-angles.

Schol. Plate II. Fig. 24.

If in a quadrilateral ABCD, the angles A, and C, which are opposite, be equal to two right-angles, then a circle may be described about that quadrilateral.

For a circle will pass through any three angles B, C, D, (as appears by 5. 4.) I say that it shall also pass thro' A, the 4th angle of such a quadrilateral: For if you deny it, let the circle pass thro' F: Therefore the right-lines BF, FD, BD being drawn, the angle  $C + F$  (a) = 2 right (b) =  $C + A$ ; wherefore A (c) is equal to F. (d) Which is absurd.

a 22. 3.

b hyp.

c 3. ax.

d 21. 1.

## PROP. XXIII. Fig. 25.

Two like and unequal segments of circles ABC, ADC, cannot be set on the same right-line AC, and on the same side thereof.

For if they are said to be like, draw the line CB cutting the circumference in D and B, join AB and AD. Because the segments are supposed like, (a) therefore is the angle  $ADC = ABC$ . (b) Which is absurd.

a 10. def. 3.

b 16. 1.

## PROP. XXIV. Fig. 26.

Like segments, of circles ABC, DEF, upon equal right-lines AC, DF, are equal one to the other. The base AC being laid on the base DF, will agree with it, because  $AC = DF$ . Therefore the segment ABC shall agree with the segment DEF (for otherwise it shall fall either within or without) and if so (a) then the segments are not like, which is contrary to the Hypothesis, and at least it shall fall partly within and partly without, and so cut in three points, (b) which is absurd. (c) Therefore the segment  $ABC = DEF$ . Which was to be demonstrated.

a 23. 3.

b 10. 3.

c 8. ax.

PROP.

PROP. XXV. Plate II. Fig. 27.

A segment of a circle ABC, being given, to describe the whole circle whereof that is a segment.

Let two right-lines be drawn AB, BC, which bisect in the points D and E. From D and E draw the perpendiculars DF, EF, meeting in the point F. I say this point shall be the center of the circle.

For the center shall be as well in (a) DF as EF, therefore it must be in the point F, which is common to them both. Which was to be done.

a cor. 1. 3.

PROP. XXVI. Fig. 28, 29.

In equal circles GABC, HDEF, equal angles stand upon equal parts of the circumference, AC, DF; whether those angles be made at the centers G, H, or at the circumferences, B, E.

Because the circles are equal, therefore is  $GA=HD$ , and  $GC=HF$ ; also by Hypothesis the angle  $G=H$ , (a) therefore  $AC=DF$ . Moreover the angle  $B(b)=\frac{1}{2}G$  (c)  $=\frac{1}{2}H(b)=E$ . (d) Therefore the segments ABC, DEF are like, and (e) consequently equal; (f) whence the remaining segments also AC, DF, are equal. Which was to be demonstrated.

a 4. 1.  
b 20. 3.  
c hyp.  
d 10. def. 3.  
e 24. 3.  
f 3. ax.

Schol. Fig. 30.

In a circle ABCD, let an arch AB be equal to DC; then shall AD be parallel to BC. For the right-line AC being drawn, the angle ACB (a)  $=CAD$ ; wherefore by 27. 1. the said sides are parallel.

a 26. 3.

PROP. XXVII. Fig. 28, 29.

In equal circles GABC, HDEF, the angles standing upon equal parts of the circumference, AC, DF, are equal between themselves, whether they be made at the centers G, H, or at the circumferences, B, E.

For if it be possible, let one of the angles AGC be  $\angle DHF$ , and make  $AGI=DHF$ ; thence is the arch AI (a)  $=DF(b)=AC$ . (c) Which is absurd.

a 26. 3.  
b hyp. 1  
c 9. ax.

Schol. Fig. 31.

A right-line EF, which being drawn from A the middle point of any periphery BC, touching the circle, is parallel

to the right-line BC, subtending the said periphery,

From the center D draw a right-line DA to the point of contact A, and join DB, DC.

a 27. 3.

b hyp.

c 4. 1.

d 10. def. 1.

e hyp.

f 28. 1.

The side GD is common, and  $DB = DC$ , and the angle BDA (a) = CDA, (because the arches BA, CA are (b) equal) therefore the angles at the base DGB, DGC are (c) equal, and (d) consequently right; but the inward angles GAE, GAF are also (e) right; (f) therefore BC, EF are parallel. Which was to be demonstrated.

PROP. XXVIII. Plate II. Fig. 28, 29.

In equal circles GABC, HDEF, equal right-lines AC, DF, cut off equal parts of the circumference, the greatest ABC, equal to the greatest DEF, and the least ALC to the least DKF.

a hyp.

b 8. 1.

c 26. 3.

d 3. ax.

From the centers G, H, draw GA, GC, and HD, HF. Because  $GA = HD$ , and  $GC = HF$ , and  $AC$  (a) =  $DF$  (b) therefore is the angle  $G = H$ ; (c) whence the arch  $ALC = DKF$ ; (d) and so the remaining arch  $ABC = DEF$ . Which was to be demonstrated.

But if the subtended line AC be  $\sqsubset$  or  $\sqsupset$  than DF, then in like manner will be the arch AC be  $\sqsubset$  or  $\sqsupset$  than DF.

PROP. XXIX. Fig. 28, 29.

In equal circles GABC, HDEF, the right-lines AC, DF, which subtend equal peripheries ABC, DEF, are equal.

a hyp.

b. 27. 3.

c 4. 1.

Draw the lines GA, GC, and HD, HF. Because  $GA = HD$ , and  $GC = HF$ , and (because the arches AC, DF are (a) equal) the angle  $G$  (b) =  $H$ , (c) therefore is the base  $AC = DF$ . Which was to be demonstrated.

This and the three precedent propositions may be understood also of the same circle.

PROP. XXX. Fig. 32.

To cut a Periphery given ABC into two equal parts.

Draw the right-line AC, and bisect it in D; from D draw a perpendicular DB meeting with the arc in B, it shall bisect the same.

a const.

b 12. ax.

c 4. 1.

d 28. 3.

For join AB, and CB. The side DB is common, and  $AD$  (a) =  $DC$ , and the angle ADB (b) =  $CD$  (c) Therefore  $AB = BC$ ; (d) whence the arch  $AB = BC$ . Which was to be done.

+

PROP



P R O P. XXXI. Plate II. Fig. 33.

In a circle the angle  $ABC$ , which is in the semicircle, is a right-angle; but the angle, which is in the greater segment  $BAC$ , is less than a right-angle, and the angle which is in the lesser segment  $BFC$  is greater than a right-angle. Moreover, the angle of the greater segment is greater than a right-angle, and the angle of the lesser segment is less than a right-angle.

From the center  $D$  draw  $DB$ . Because  $DB = DA$ , therefore is the angle  $A(a) = DBA$ , and the angle  $DCB(a) = DBC$ ,  $(b)$  therefore the angle  $ABC = A + ACB$ ,  $(c) = EBC$ ,  $(d)$  so that  $ABC$  and  $EBC$  are right-angles. *W. W. D.*  $(e)$  Therefore  $BAQ$  is an acute angle. *W. W. D.* And further, whereas  $BAC + BFC(f) = 2$  right-angles, therefore  $BFC$  is an obtuse angle. Lastly, the angle contained under the right-line  $CB$ , and the arch  $BAC$  is greater than the right-angle  $ABC$ ; but the angle made by the right-line  $CB$ , and the periphery of the lesser segment  $BFC(g)$  is less than the right-angle  $EBC$ . Which was to be demonstrated.

*Schol.*

In a right-angled triangle  $ABC$ , if the hypotenuse (or line subtending the right-angle)  $AC$  be bisected in  $D$ , a circle drawn from the center  $D$ , through the point  $A$ , shall also pass through the point  $B$ ; as you may easily demonstrate from this prop. and 21. 1.

P R O P. XXXII. Fig. 34.

If a right-line  $AB$  touch a circle, and from the point of contact be drawn a right-line  $CE$ , cutting the circle, the angles  $ECB$ ,  $ECA$ , which it makes with the tangent line, are equal to those angles  $EDC$ ,  $EFC$ , which are made in the alternate segments of the circle.

Let  $CD$ , the side of the angle  $EDC$  be perpendicular to  $AB$ ,  $(a)$  for it is to the same purpose,  $(b)$  therefore  $CD$  is the diameter,  $(c)$  therefore the angle  $CED$  in a semicircle is a right-angle,  $(d)$  and therefore the angle  $D + DCE =$  to a right-angle  $(e) = ECB + DCE$ ,  $(f)$  Therefore the angle  $D = ECB$ . Which was to be dem.

Now whereas the angle  $ECB + ECA(g) = 2$  right-angles  $(b) = D + F$ , from both of these take away  $ECB$  and  $D$ , which are equal,  $(h)$  there remains  $ECA = F$ . Which was to be dem.

P R O P.

a 5. 1.  
b 2. ax.  
c 32. 1.  
d 10. def. 1.  
e cor. 17. 1.  
f 22. 3.

g 9. ax.

a 26. 3.  
b 19. 3.  
c 31. 3.  
d 32. 1.  
e const.  
f 3. ax.  
g 13. 1.  
h 22. 3.  
k 3. ax.

## P R O P. XXXIII. Plate II. Fig. 35.

Upon a right-line AB to describe a segment of a circle AIEB which shall contain an angle AIB, equal to a right-lined angle given C.

a 23. 1.

c constr.

c 6. 1.

d cor. 16. 3.

a 32. 13.

f constr.

(a) Make the angle  $BAD = C$ . Through the point A draw the line AE perpendicular to HD. At the other end of the line given AB make an angle  $ABF = BAF$ , one of the sides of which shall cut the line AE in F; from the center F, through the point A, describe a circle, which shall pass through B. (Because the angle  $FBA$  (b)  $= FAB$ , and (c) therefore  $FB = FA$ ) AIB is the segment sought. For because HD is perpendicular to the diameter AE, therefore HD (d) touches the circle which AB cuts. And therefore the angle AIB (e)  $= BAD$  (f)  $= C$ . Which was to be done.

## P R O P. XXXIV. Fig. 36.

From a circle given ABC to cut off a segment ABC containing an angle B, equal to a right-lined angle given D.

a 17. 3.

b 23. 1.

c 32. 3.

d constr.

(a) Draw a right-line EF which shall touch the circle given in A, (b) let AC be drawn making an angle  $FAC = D$ . Also draw (b) AB, making the angle  $BAE = D$ , and join B, C. The line AB shall cut off the segment ABC containing an angle B (c)  $= CAF$  (d)  $= D$ . Which was to be done.

## P R O P. XXXV. Fig. 37.

If in a circle DBCA two right-lines AB, DC cut each other, the rectangle comprehended under the segments AE, EB, of the one, shall be equal to the rectangle comprehended under the segments CE, ED of the other.

1. Case. If the right-lines cut one the other in the center, the thing is evident.

a 5. 2.

b sch. 48. 1.

c 47. 1.

d hyp.

e 3. ax.

2. Case. If one line AB (Fig. 38.) passes thro' the center F, and bisects the other line CD, then draw FD. Now the rectangle  $AEB + FEq$  (a)  $= FBq$ . (b)  $= FDq$ . (c)  $= EDq + FEq$  (d)  $= CED + FEq$ . (e) Therefore the rectangle  $AEB = CED$ . Which was to be demonstrated.

3. Case. If one of the lines AB (Fig. 39.) be the diameter, and cut the other line CD unequally, bisect CD by FG, a perpendicular from the center.

These

These are equal.  $\left\{ \begin{array}{l} \text{The rectangle AEB} + \text{FEq.} \\ (f) \text{ FBq (FDq)} \\ (g) \text{ FGq} + \text{GDq} \\ \text{FGq} + (h) \text{ GEq} + \text{Rectang. CED.} \\ (k) \text{ FEq} + \text{CED.} \end{array} \right.$

f 5. 2.  
g 47. 1.  
h 5. 2.  
k 47. 1.  
l 3. ax.

Fig. 40.

Fig. 41.  
a 15. 1.  
b 21. 3.  
c cor. 32. 1.  
d 4. 6.  
e 16. 6.

a 18. 3.  
b 47. 1.  
c 6. 2.  
d 3. ax.

Fig. 43.  
a 3. 3.

b 47. 1.  
c 6. 2.  
d 47. 1.  
e 3. ax.  
Fig. 44.  
a 32. 3.  
b 32. 1.  
c 4. 6.  
d 17. 6.

(l) Therefore the rectangle AEB = CED.

4. Case. If neither of the right-lines AB, CD pass thro' the center, then through the point of intersection E, draw the diameter GH. By that which hath been already demonstrated it appears, that the rectangle AEB = GEH = CED. Which was to be demonstrated.

More easily, and generally, thus; join AC and BD, then because the angles (a) CEA, DEB, and (b) also C, B (upon the same arch AD) are equal, thence are the triangles CEA, BED, (c) equiangular. (d) Wherefore CE : EA :: EB : ED, and (e) consequently CE x ED = AE x EB. Which was to be demonstrated.

The citations out of the 6 Book, both here and in the following prop, have no dependance on the same; so that it was free to use them.

PROP. XXXVI. Plate II. Fig. 42.

If any point be taken without a circle EBC, and from that point two right-lines DA, DB, fall upon the circle, whereof one DA cuts the circle, the other DB touches it, the rectangle comprehended under the whole line DA that cuts the circle, and DC, that part which is taken from the point given D to the convex of the periphery, shall be equal to the square made of the tangent line.

1. Case. If the secant AD passes thro' the center, then join EB, this (a) will make a right-angle with the line DB, wherefore DBq + (d) EBq (ECq,) (b) = EDq (c) = AD x DC + ECq. Therefore AD x DC = DBq. Which was to be demonstrated.

2. Case. But if AD passes not thro' the center, then draw EC, EB, ED, and EF perpendicular to AD, (a) wherefore AC is bisected in F.

Because BDq + EBq (b) = DEq (b) EFq + FDq (c) = EFq + ADC + FCq (d) = ADC + CEq (EBq.) (e) Therefore is BDq = ADC. Which was to be demonstrated.

More easily, and generally thus; draw AB and BC. Then, because the angles A, and DBC (a) are equal, and the angle D common to both; thence are the triangles BDC, ADB (b) equiangular. (c) Wherefore AD : DB :: DB : CD; and (d) consequently AD x DC = DBq: Which was to be demonstrated.

D

Coroll.



Coroll. Plate II. Fig. 45.

1. Hence, If from any point A, taken without a circle, there be several lines AB, AC drawn which cut the circle; the rectangle comprehended under the whole lines AB, AC, and the outward parts AE, AF, are equal between themselves.

a 36. 3. For if the tangent AD be drawn, then is  $CAF = ADq$  (a) = BAE.

2. It appears also from hence, that if two lines AB, AC, (Fig. 46.) drawn from the same point do touch a circle, those two lines are equal one to the other.

b 36. 3. For if AE be drawn cutting the circle, then is  $ABq$  (a) = EAF (b) = ACq,

3. It is also evident, that from a point A, taken without a circle, there can be drawn but two lines AB, AC, that shall touch the circle.

c 2. cor. For if a third line AD be said to touch the circle, thence is  $AD(c) = AB(c) = AC$ . (d) Which is absurd.

d 8. 3. 4. And, on the contrary, it is plain, that if two equal right-lines AB, AC, fall from any point A, upon the convex periphery of a circle, and that if one of these equal lines AB touch the circle, then the other AC touches the circle also.

e 2. cor. For if possible, let not AC, but another line AD, touch the circle; therefore is  $AD(e) = AC(f) = AB$ . f hyp. (g) Which is absurd.

g 8. 3.

PROP. XXXVII. Fig. 47.

If without a circle EBF any point D be taken, and from that point two right-lines DA, DB fall on the circle, whereof one line DA cuts the circle, the other DB falls upon it; and if also the rectangle comprehended under the whole line that cuts the circle, and under that part of it DC which is taken betwixt the point D and the convex periphery, be equal to that square which is made of the line DB falling on the circle, I say that that line DB so falling shall touch the circle given.

a 17. 3. From the point D (a) let a tangent DF be drawn, and b hyp. from the center E draw ED, EB, EF. Now because c 36. 3.  $DBq(b) = ADC(c) = DFq$ , therefore is  $DB(d) = DF$ : d 1. ax. & But  $EB = EF$ , and the side ED common; (e) therefore fcb. 4. 1. the angle  $EBD = EFD$ ; but EFD is a right-angle; and e 8. 1. (f) therefore EBD is right also; and (g) therefore DB touches the circle. Which was to be dem.

f 12. ax.

g cor. 16. 3.

h 8. 1.

Coroll.

From hence it follows that the (h) angle  $EDB = EDF$ .

The End of the third Book.

The FOURTH BOOK  
OF  
EUCLID'S  
ELEMENTS.

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*Definitions.*

I. **A** Right-lined figure is said to be inscribed in a right-lined figure, when every one of the angles of the inscribed figure touch every one of the sides of the figure wherein it is inscribed.

*So the triangle DEF is inscribed in the triangle ABC.*  
Plate II. Fig. 48.

II. In like manner a figure is said to be described about a figure, when every one of the sides of the figure circumscribed, touch every one of the angles of the figure about which it is circumscribed.

*So the triangle ABC is described about the triangle DEF.*

III. A right-lined figure is said to be inscribed in a circle, when all the angles of that figure which is inscribed touch the circumference of the circle. As Fig. 50.

IV. A right-lined figure is said to be described about a circle, when all the sides of the figure which is circumscribed touch the periphery of the circle. As Fig. 49.

V. After the like manner a circle is said to be inscribed in a right-lined figure, when the periphery of the circle touches all the sides of the figure, in which it is inscribed. Fig. 49.

VI. A circle is said to be described about a figure when the periphery of the circle touches all the angles of the figure, which it circumscribes.

VII. A right-line is said to be fitted or applied in a circle when the extremes thereof fall upon the circumference; as the right-line AB. Fig. 6.

PROP. I. Fig. 51.

*In a circle given ABC to apply a right-line AB equal to a right-*

right-line given  $D$ , which doth not exceed  $AC$ , the diameter of the circle.

a 3. *post.* 5

3. 1.

b 15. *def.* 1.

c *constr.*

From the center  $A$  at the distance  $AE = D$  ( $a$ ) describe a circle meeting with the circle given in  $B$ , draw  $AB$ . Then is  $AB$  ( $b$ )  $= AE$  ( $c$ )  $= D$ . Which was to be done.

P R O P. II. Plate II. Fig. 49, 50.

In a circle given  $ABC$  to describe a triangle  $ABC$ , equiangular to a triangle given  $DEF$ .

a 17. 3.

b 23. 1.

c 32. 3.

d *constr.*

e 32. 1.

Let the right-line  $GH$  ( $a$ ) touch the circle given in  $A$ ; ( $b$ ) make the angle  $HAC = E$ , ( $b$ ) and the angle  $GAB = F$ , then join  $BC$ ; and the thing is done.

For the angle  $B$  ( $c$ )  $= HAC$  ( $d$ )  $= E$ , and the angle  $C$  ( $c$ )  $= GAB$  ( $d$ )  $= F$ ; ( $e$ ) whence also the angle  $BAC = D$ . Therefore the triangle  $BAC$  inscribed in the circle is equiangular to  $DEF$ . Which was to be done.

P R O P. III. Fig. 52, 53.

About a circle given  $IABC$  to describe a triangle  $LNM$ , equiangular to a triangle given  $DEF$ .

a 23. 1.

b 17. 3.

c 13. *ax*

d 11. 3.

Produce the side  $EF$  on both sides; at the center  $I$  ( $a$ ) make an angle  $AIB = DEG$ , and an angle  $BIC = DFH$ . Then through the points  $A, B, C$ , let three right-lines  $LN, LM, NM$ , ( $b$ ) be drawn, touching the circle, and the thing is done.

For it's evident that the right-lines  $LN, LM, MN$ , will meet and make a triangle, ( $c$ ) because the angles  $LAI, LBI$  are right; so that if the ( $d$ ) right-line  $AB$  was drawn, it would make the angles  $LAB, LBA$ , less than two right-angles.

e *sch.* 32. 1.

f 13. *ax*

g *constr.*

h 3. *ax*

k 32. 1.

Since therefore the angle  $AIB + L$  ( $e$ )  $= 2$  right-angles ( $f$ )  $= DEG + DEF$ , and  $AIB$  ( $g$ )  $= DEG$ ; ( $b$ ) therefore is the angle  $L = DEF$ . By the same way of reasoning the angle  $M = DFE$ . ( $k$ ) Therefore also the angle  $N = D$ . And therefore the triangle  $LMN$ , described about the circle, is equiangular to  $DEF$ , the triangle given. Which was to be done.

P R O P. IV. Fig. 54.

In a triangle given  $ABC$ , to describe a circle  $EFG$ .

a 9. 1.

b 12. 1.

( $a$ ) Bisect the angles  $B$  and  $C$  with the right-lines  $BD, CD$ , meeting in the point  $D$ , ( $b$ ) and draw the perpendiculars  $DE, DF, DG$ . A circle described from the center



center D through E, will pass through G and F, and touch the three sides of the triangle.

For the angle DBE (c) = DBF; and the angle DEB (d) = DFB; and the side DB common, (e) therefore DE = DF. For the same reason DG = DF, The circle therefore described from the center D passes through the three points E, F, G; and whereas the angles at E, F, G, are right, therefore it touches all the sides of the triangle. Which was to be done.

c constr.  
d 12 ax,  
e 26. 1.

Schol.

Hence, The sides of a triangles being known, their segments which are made by the touching of the circle inscribed, shall be found, Thus;

Pet. Herig.

Let AB be 12, AC 18, BC 16, then is  $AB + BC = 28$ . Out of which subduct  $18 = AC = AE + FC$ , there remains  $10 = BE + BF$ . Therefore BE, or BF = 5; and consequently FC, or CG = 11. Wherefore GA, or AE, = 7.

P R O P. V. Plate II. Fig. 55, 56.

About a triangle given ABC, to describe a circle FABC.

(a) Bisect any two sides BA, CA, with perpendiculars DF, EF, meeting in the point F. I say this shall be the center of the circle.

a 10. 3  
11. 1.

For, let the right-lines FA, FB, FC be drawn. Now because AD (b) = DB, and the side DF common, and the angles FDA (c) = FDB, therefore is FB (d) = FA. After the same manner is FC = FA. Therefore a circle described from the center F shall pass through the angles of the triangle given (viz.) B, A, C. Which was to be done.

b constr.  
c constr. 3  
12. ax.  
d 4. 1.

Coroll.

\* Hence, if a triangle be acute-angled, the center shall fall within the triangle; if right-angled, in the side opposite to the right-angle, and if obtuse-angled, without the triangle.

31. 3.

Schol.

By the same method may a circle be described, that shall pass through three points given, not being in the same strait-line.

D 3

P R O P.

## PROP. VI. Plate II. Fig. 57.

In a circle given EABCD to inscribe a square ABCD.

a 11. 1.

(a) Draw the diameters AC, BD cutting each other at right-angles in the center E. Join the extremes of these diameters with the right-lines AB, BC, CD, DA. And the thing is done.

b 26. 3.

c 29. 3.

d 31. 3.

e 29. def. 1.

Now because the four angles at E are right, the (b) arches and (c) subtended lines AB, BC, CD, DA, are equal; therefore is the figure ABCD equilateral, and all the angles in semicircles, and so (d) right. (e) Therefore ABCD is a square inscribed in a circle given. Which was to be done.

## PROP. VII. Fig. 58.

About a circle given EABCD, to describe a square FHIG.

a 17. 3.

Draw the Diameters AC, BD, cutting one the other at right-angles; through the extremes of these diameters (a) draw tangents meeting in F, H, I, G. And the thing is done.

b 18. 3.

c 28. 1.

d 34. 1.

e 15. def. 1.

f 10. def. 1.

For because (b) the angles A and C are right, (c) therefore is FG parallel to HI. After the same manner is FH parallel to GI, and therefore FHIG is a Pgr. and also right-angled. It is equilateral because  $FG = HI = DB = CA = FH = GI$ .

Wherefore PHIG is a (f) square circumscribed to the circle given. Which was to be done.

## Schol. Fig. 59.

A square ABCD described about a circle is double of the square EFGH inscribed in the same circle.

For the rectangle  $HB = 2 HEF$  and  $HD = 2 HGF$ , by the 41. 1.

## PROP. VIII. Fig. 58.

In a square given FGHI, to inscribe a circle EABCD.

a 7. ax.

b hyp.

c 33. 1.

Bisect the sides of the square in the points B, D, A, C, cutting one the other in E, a circle drawn from the center E, thro' A, shall be inscribed in the square.

For because FA and HC are (a) equal and (b) parallel, (c) therefore is FH parallel to AC, parallel to GL. After the same manner is FG parallel to BD, parallel to HI; therefore EF, EG, EH, EI, are parallelograms. There-

Therefore  $FA(d) = FB(e) = AE = BE = CE = ED$ . The circle therefore described from the center E, through A shall pass through A, B, C, D, and touch the sides of the square, since the angles A, B, C, D, are right. *Which was to be done.*

d 7. ax.  
c 34. 1.

P R O P. IX. Plate II. Fig. 57.

*About a square given ABCD, to describe a circle EABCD.*  
Draw the diagonals AC, BD, cutting one the other in E. From the center E through A describe a circle; I say this circle is circumscribed to the square.

For the angles ABD and BAC are (a) half of right-angles; (b) therefore  $EA = EB$ . After the same manner is  $EA = ED = EC$ . The circle therefore described from the center E passes through A, B, C, D the angles of the square given. *Which was to be done.*

a 4. cor.  
32. 1.  
b 6. 1.

P R O P. X. Fig. 60.

*To make an Isosceles triangle ABD, having each angle at the base B, and ADB double to the remaining angle A.*

Take any right-line AB, and divide it in C, (a) so that  $AB \times BC$  may be equal to  $AC^2$ . From the center A thro' B, describe the circle ABD; and in this circle (b) apply  $BD = AC$ , and join AD; I say ABD is the triangle required.

a 11. 2.

b 1. 4.

For draw DC, and through the points C, D, A, (c) draw a circle. Now because  $AB \times BC = AC^2 = BD^2$ , (d) it is evident that BD touches the circle ACD which CD cutteth; (e) therefore is the angle  $BDC = A$ , and therefore the angle  $BDC + CDA$  (f)  $= A + CDA$  (g)  $= BCD$ . But  $BDC + CDA = BDA$  (h)  $= CBD$ ; (i) therefore the angle  $BCD = CBD$ , and therefore DC (j)  $= DB = (m)$  AC; (n) wherefore the angle  $CDA = A = BDC$ , therefore  $ADB = 2A = ABD$ . *Which was to be done.*

c 5. 4.  
d 37. 3.  
e 32. 3.  
f 2. ax.  
g 32. 1.  
h 5. 1.  
k 1. ax.  
l 6. 1.  
m constr.  
n 5. 1.

Coroll.

Whereas all the angles A, B, D, (b) make up two right-angles, it's evident that A is one fifth of two right-angles.

h 32. 1.

P R O P. XI. Fig. 61, 62,

*In a circle given ABCDE to inscribe a Pentagon ABCDE equilateral and equiangular.*

D 4

(a) Describe



a 10. 4.

b 2. 4.

c 9. 1.

d 26. 3.

e 29. 3.

f 27. 3.

g 2. ax.

(a) Describe an Iſoſceles triangle FGH, having each angle at the baſe double to the other; to the circle, (b) inſcribe a triangle CAD equiangular to the ſaid triangle FGH. (c) Biſect the angles at the baſe ACD and ADC with the right-lines DB, CE meeting with the circumference in B and E, join the right-lines CB, BA, AE, ED. *Then I ſay it is done.*

For it is evident by conſtruction, that the angles CAD, CDB, BDA, DCE, ECA, are equal; wherefore the (d) arches and (e) the lines ſubtending them DC, CB, BA, AE, DE, are equal. Therefore the pentagon is equilateral, and equiangular, (f) becauſe the angles of it BAE, AED, &c. ſtand on equal (g) arches BCDE, ABCD, &c.

A more eaſy practice of this problem ſhall be deliver'd at 10. 13.

*Coroll.*

Hence, each angle of an equilateral and equiangular pentagon is equal to three-fifths of two right-angles, or ſix-fifths of one right-angle.

*Schol. Fig. 63.*

*Pet. Herig.*

Generally all figures whoſe number of ſides is odd, are inſcribed in circles by the help of Iſoſceles triangles, whoſe angles at the baſe are multiples of thoſe at the top: and figures whoſe number of ſides is even, are inſcribed in a circle by the help of Iſoſceles triangles, whoſe angles at the baſe are multiples ſeſquialter of thoſe at the top.

As in the Iſoſceles triangle CAB if the angle  $A = 3C = B$ , then will AB be the ſide of a Heptagon. If  $A = 4C$ , then is AB the ſide of an Enneagon. But if  $A = \frac{1}{2}C$ , then is AB the ſide of a ſquare. And if  $A = 2\frac{1}{2}C$  CAB will ſubtend the ſixth part of the circumference, and likewise if  $A = 3\frac{1}{2}C$  then will AB be the ſide of an Octagon.

P R O P. XII. Plate III. Fig. 1.

*About a circle given FABCDE, to deſcribe a pentagon HIKLG, equilateral and equiangular.*

a 11. 4.

(a) Inſcribe a pentagon ABCDE in the circle given; and from the center draw the right-lines FA, FB, FC, FD, FE; and to thoſe lines draw ſo many perpendiculars GAH, HBI, ICK, KDL, LEG, meeting in the points H, I, K, L, G, and the thing is done. For be-

cauſe

cause GA, GE from the same point G (*b*) touch the circle, (*c*) therefore is  $GA=GE$ , and (*d*) therefore the angle  $GFA=GFE$ , therefore the angle  $AFE=2\ GFA$ . After the same manner is the angle  $AFH=HFB$ , and consequently the angle  $AFB=2\ AFH$ . (*e*) But the angle  $AFE=AFB$ , (*f*) therefore the angle  $GFA=AFH$ . But also the angle  $FAH$  (*g*)  $=FAG$ , and the side FA is common, (*b*) therefore  $HA=AG=GE=EL$ , &c. (*k*) Therefore HG, GL, LK, KI, IH, the sides of the pentagon are equal, and so also are the angles, because double of the equal angles AGF, AHF, therefore, &c.

*Coroll.*

After the same manner, if any equilateral and equi-angled figure be described in a circle, and at the extreme points of the semi-diameters, drawn from the center to the angles, be drawn perpendicular lines to the said diameters; I say, that these perpendiculars shall make another figure of as many equal sides and equal angles, circumscribed to the circle.

PROP. XIII. Plate III. Fig. 2.

In an equilateral and equiangular pentagon given ABCDE so inscribe a circle FGHK.

(*a*) Bisect two angles of the pentagon A and B, with the right-lines AI, BK, meeting in the point F. From F draw the perpendiculars FG, FH, FI, FK, FL. Then a circle described from the center F through G will touch all the sides of the pentagon.

Draw FC, FD, FE. Because BA (*b*)  $=BC$ , and the side BF common, and the angle  $FBA$  (*c*)  $=FBC$ , (*d*) therefore is  $AF=FC$ , and the angle  $FAB=FCB$ , but the angle  $FAB$  (*e*)  $=\frac{1}{2}\ BAE=\frac{1}{2}\ BCD$ . Therefore the angle  $FCB=\frac{1}{2}\ BCD$ . After the same manner are all the whole angles C, D, E bisected. Now whereas the angle  $FGB$  (*f*)  $=FHB$ , and the angle  $FBH=FBG$ , and the side FB is common, (*g*) therefore is  $FG=FH$ . In like manner are all the right-lines FH, FI, FK, FL, FG equal. Therefore a circle described from the center F, through G, passes through the points H, I, K, L, and (*b*) touches the side of the pentagon, because the angles at those points are right. Which was to be done.

*Coroll.*

Hence, if any two nearest angles of an equilateral and equiangular figure are bisected, and from that point in which,

b cor. 16.3.

c 2. cor. 36.

d 8. 1.

e 27. 3.

f 7. ax.

g 12. ax.

h 26. 1.

k 2. ax.

a 9. 1.

b hyp.

c constr.

d 4. 1.

e hyp.

f 12. ax.

g 26. 1.

h cor. 16.3.

which the lines meet that bisect the angles, be drawn right-lines to the remaining angles of the figure, all the angles of the figure shall be bisected.

*Schol.*

By the same method may a circle be inscribed in any equilateral and equiangular figure.

P R O P. XIV. Plate III. Fig. 1.

*About a pentagon given ABCDE, equilateral and equiangular, to describe a circle FABCDE.*

Bisect any two angles of the pentagon with the right-lines AF, BF, meeting in the point F; the circle described from the center F through A shall be described about the pentagon.

a cor. 13. 4.  
b 6. 1.

For let FC, FD, FE be drawn. (a) Then the angles C, D, E are bisected; (b) and therefore FA, FB, FC, FD, FE are equal; therefore the circle described from the center F passes through A, B, C, D, E, all the angles of the pentagon. Which was to be done.

*Schol.*

By the same method is a circle described about any figure which is equilateral and equiangular.

P R O P. XV. Fig. 3.

*In a circle given GABCDEF to inscribe an Hexagon (or six-sided figure) ABCDEF equilateral and equiangular.*

Draw the diameter AD; from the center D through the center G describe a circle cutting the circle given in the points C and E. Draw the diameters CF, EB; and join AB, BC, CD, DE, EF, FA. Then I say it's done.

a cor. 32. 1.

b 15. 1.

c cor. 13. 1.

d 26. 3.

e 29. 3.

f 27. 3.

For the angle CGD (a)  $= \frac{1}{2}$  of 2 right-angles (a)  $=$  DGE (b)  $=$  AGF (b)  $=$  AGB. (c) Therefore BGC  $= \frac{1}{2}$  of 2 right-angles  $=$  FGE; therefore the (d) arches and (e) subtenses AB, BC, CD, DE, EF, are equal. Therefore the hexagon is equilateral; but it is equiangular also, (f) because all the angles of it stand upon equal arches.

*Corol.*

1. Hence, the side of an hexagon inscribed in a circle is equal to the semidiameter.

2. Hereby



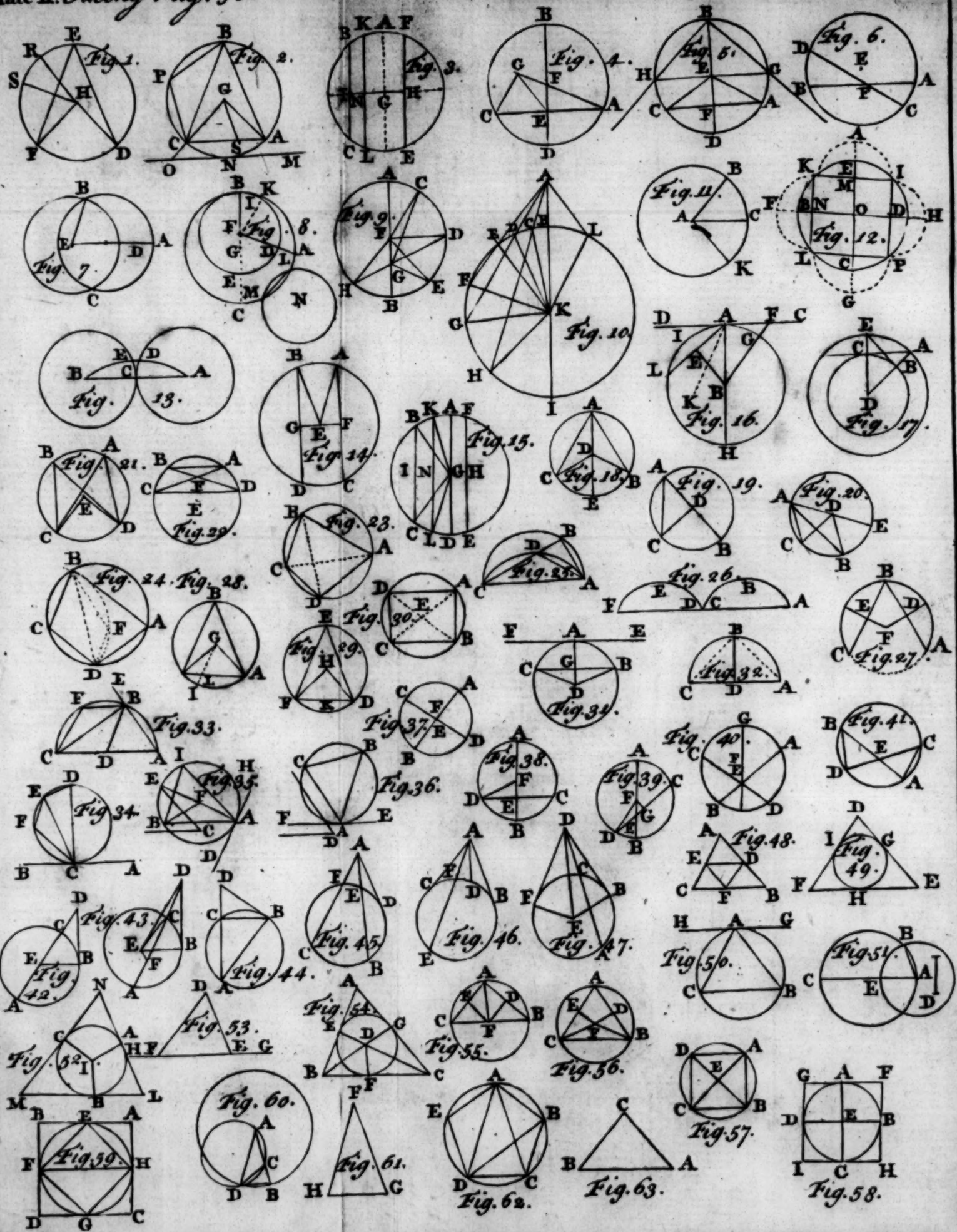
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2. Hereby an equilateral triangle ACE may very easily be described in a circle given.

*Schol. Probl.*

To make a true hexagon upon a right-line given CD.

(a) Make an equilateral triangle CGD upon the line given CD; from the center G through C and D describe a circle. That circle shall contain the hexagon made upon the given line CD.

*Andr.*

*Tacq.*

2 1. 1.

P R O P. XVI. Plate III. *Fig. 4.*

In a circle given AEBC, to inscribe a quindecagon (or fifteen-sided figure) equilateral and equiangular.

(a) Inscribe an equilateral pentagon AEFGH in the circle given, and (b) also an equilateral triangle ABC, then I say BF is the side of the quindecagon required.

a 11. 4.

b 2. 4.

For the arch AB (c) is  $\frac{1}{3}$  or  $\frac{1}{15}$  of that periphery whereof AF is  $\frac{2}{3}$  or  $\frac{6}{15}$ , therefore the remaining part BF is  $\frac{1}{15}$  of the periphery; and therefore the quindecagon, whose side is BF, is equilateral; but it is equiangular also (d) because all the angles stand on equal arches of a circle, whereof every one  $\frac{1}{15}$  of the whole circumference. Therefore, &c.

c constr.

d 27. 3.

*Schol.*

A circle is geometrically divided into parts

|                                                                                                                                                                                                                                                                               |                                     |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------|
| $\left\{ \begin{array}{l} 4, 8, 16, \&c. \text{ by } 6, 4, \text{ and } 9, 1. \\ 3, 6, 12, \&c. \text{ by } 15, 4, \text{ and } 9, 1. \\ 5, 10, 20, \&c. \text{ by } 11, 4, \text{ and } 9, 1. \\ 15, 30, 60, \&c. \text{ by } 16, 4, \text{ and } 9, 1. \end{array} \right.$ | 4, 8, 16, &c. by 6, 4, and 9, 1.    |
|                                                                                                                                                                                                                                                                               | 3, 6, 12, &c. by 15, 4, and 9, 1.   |
|                                                                                                                                                                                                                                                                               | 5, 10, 20, &c. by 11, 4, and 9, 1.  |
|                                                                                                                                                                                                                                                                               | 15, 30, 60, &c. by 16, 4, and 9, 1. |

Any other way of dividing the circumference into any parts given is as yet unknown; wherefore in the construction of ordinate figures, we are forced to have recourse to mechanic artifices, concerning which you may consult the writers of practical Geometry.

*The End of the fourth Book.*

The

# The FIFTH BOOK OF EUCLIDE ELEMENTS.

## Definitions.

I. **A** Part, is a magnitude of a magnitude, a less of a greater, when the less measureth the greater.

II. Multiple, is a greater magnitude in respect of a lesser, when the lesser measureth the greater.

III. Ratio, is the mutual habitude or respect of two magnitudes of the same kind each to other, according to quantity.

*In every ratio that quantity which is referr'd to another quantity, is called the antecedent of the ratio, and that which the other is referr'd, is called the consequent of the ratio, as in the ratio of 6 to 4, 6 is the antecedent and 4 the consequent.*

Note, The quantity of any ratio is known by dividing the antecedent by the consequent; as the ratio of 12 to 5 is expressed by  $1\frac{2}{5}$ ; or the quantity of the ratio of A to B is

Wherefore, often for brevity sake we denote the quantities of ratio's thus;  $\frac{A}{B}$ , or  $\frac{C}{D}$ , or  $\frac{A}{B} = \frac{C}{D}$ , that is, the ratio of A to B is greater, equal, or less than the ratio of C to D. Which must be well observed by those who would understand this Book.

Concerning the divers species of ratio's, you may please to consult interpreters.

IV. Proportion is a similitude of ratio's.

That which is here termed proportion, is more properly called proportionality or analogy: for proportion commonly denotes no more than the ratio betwixt two magnitudes.

V. The

V. Those numbers are said to have a ratio betwixt them, which being multiplied may exceed one the other.

E, 12 | A, 4. B, 6. | G, 24.  
F, 30 | C, 10 D, 15. | H, 60. VI. Magnitudes are said to be in the same ratio, the first A, to the second B, and the third C, to the fourth D, when the equimultiples E and F of the first A, and the third C, compared with the equimultiples G, H, of the second B, and the fourth D, according to any multiplication whatsoever, either both together E, F are less than G, H both together, or equal taken together, or exceed one the other together, if those be taken E, G, and F, H, which answer one to the other.

The note hereof is : : ; as A : B : : C : D. That is, as A is to B, so is C to D, which signifies that A to B, and C to D, are in the same ratio. We sometimes thus express it

$$\frac{A}{B} = \frac{C}{D}, \text{ that is, } A : B : : C : D.$$

VII. Magnitudes that have the same ratio (A : B : : C : D), are called proportional.

VIII. When of equimultiples, E, the multiple of the first magnitude A, exceeds G, the multiple of the second B, but F, the multiple of the third C, exceeds not H, the multiple of the fourth D, then the first A to the second B, has a greater ratio than the third C to the fourth D.

If  $\frac{A}{B} > \frac{C}{D}$ , it is not necessary from this definition, that E should always exceed G, when F is less than H; but it is granted that this may be.

IX. Proportion consists in three terms at least. Whereof the second supplies the place of two.

X. When three magnitudes A, B, C, are proportional, the first A is said to have a duplicate ratio to the third C, of that it hath to the second B: But when four magnitudes A, B, C, D are proportional, the first A is said to have a triplicate ratio to the fourth D, of what it has to the second B; and so always in order one more, as the proportion shall be extended.

Duplicate ratio is thus expressed  $\frac{A}{C} = \frac{A}{B}$  twice, that is, the ratio of A to C is duplicate of the ratio of A to B. Triplicate



Triplicate ratio is thus expressed;  $\frac{A}{D} = \frac{A}{B}$  thrice; that

is, the ratio of A to D is triplicate of the ratio of A to B.

$\therefore$  Denotes continued proportionals; as A, B, C, D,  $\therefore$  also 2, 6, 18, 54,  $\therefore$  are continual proportionals.

XI. Homologous, or Magnitudes of like ratio, are antecedents to antecedents, and consequents to consequents; that is, if  $A : B :: C : D$ ; A and C; as well as B and D are called homologous magnitudes.

XII. Alternate proportion is the comparing of antecedent to antecedent, and consequent to consequent. As if  $A : B :: C : D$ . therefore alternately, or by permutation, As  $A : C :: B : D$ . by the 16. of 5.

In this definition, and the five following, names are given to the six ways of arguing which are often used by Mathematicians: the force of which inferences depends on the propositions of this Book, which are named in their explications.

XIII. Inverse ratio is when the consequent is taken as the antecedent, and so compared to the antecedent as the consequent; as  $A : B :: C : D$ ; therefore inversely  $B : A :: D : C$ . by cor. 4. 5.

XIV. Compounded ratio is when the antecedent and consequent taken both as one, are compared to the consequent it self. As  $A : B :: C : D$ ; therefore by composition  $A + B : B :: C + D : D$ , by 18. 5.

XV. Divided ratio is when the excess wherein the antecedent exceedeth the consequent, is compared to the consequent. As  $A : B :: C : D$ ; therefore by division,  $A - B : B :: C - D : D$ , by 17. 5.

XVI. Converse ratio is when the antecedent is compared to the excess wherein the antecedent exceeds the consequent. As  $A : B :: C : D$ ; therefore by converse ratio  $A : A - B :: C : C - D$ , by the coroll. of the 19 of the 5.

XVII. Proportion of equality is where there are taken more magnitudes than two in one order, and also as many magnitudes in another order, comparing two to two being in the same ratio; it follows that as in the first order of magnitudes, the first is to the last, so in the second order of magnitudes, is the first to the last. Or otherwise: it is comparison of the extremes together, the mean magnitudes being omitted.

Thus let A, B, C, be three magnitudes and D, E, F, three others, and taking them two by two, let them be in the same proportion, that is, let  $A : B :: D : E$ , and  $B : C :: E : F$ ; now if it be inferr'd that A, the first of the first order

er, is to C the last, as D the first of the second order, is to F the last, this form of arguing is said to be *ex æquo*, or from equality.

XVIII. Ordinate proportion is, when antecedent is to consequent, as antecedent to consequent, and as the consequent is to any other, so is the consequent to any other. *As when*  $A:B::D:E$ . *also*  $B:C::E:F$ . *and then it shall be*  $A:C::D:F$ , *by the 22. of the 5.*

XIX. Perturbate proportion is, when three magnitudes being put, and others also, which are equal to these in multitude, as in the first magnitudes the antecedent is to the consequent, so in the second magnitudes is the antecedent to the consequent: and as in the first magnitudes the consequent is to any other, so in the second magnitudes is any other, to the antecedent. *Thus if* A, B, C, *and* E, F, G, *are two sets of magnitudes, if* A *the first of the first set, is to* B *the second, as* F *the second of the second set, is to* G *the last; and also if* B *the second of the first set is to* C *the last, as* E *the first of the second set is to* F *the second, such is called perturbate proportion, and by the 23. 5.*  $A:C::E:G$ .

XX. Any number of magnitudes being put; the proportion of the first to the last is compounded out of the proportions of the first to the second, the second to the third, and the third to the fourth, &c. to the last.

Let there be any number of magnitudes A, B, C, D,

by this definition  $\frac{A}{D} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D}$ .

*Axiom.*

Equimultiples of the same, or of equal magnitudes are equal to each other.

PROP. I. Plate III, Fig. 5.

*If there be any number of magnitudes* AB, CD, *equimultiples to a like number of magnitudes* E, F, *each to each; what ever multiple one magnitude* AB *is of one* E, *the same multiple is all the magnitudes* AB+CD *to all the other magnitudes* E+F.

Let AG, GH, HB, the parts of the quantity AB, be equal to E, and also let CI, IK, KD, the parts of the quantity CD, be equal to F. The Number of these are put equal to those. Now whereas  $AG+CI (a) = E+F$ ;

2 2. ax.

a 2. ax.

F; (a) and  $GH + IK = E + F$ ; (a) and  $HB + KD = E + F$ , it is evident that  $AB + CD$  doth so often contain  $E + F$  as one  $AB$  contains  $E$ , Which was to be done.

## P R O P. II. Plate III. Fig. 6.

If the first  $AB$  be the same multiple of the second  $C$ , as the third  $DE$  is of the fourth  $F$ , and if the fifth  $BG$  be the same multiple of the second  $C$ , as the sixth  $EH$  is of the fourth  $F$ ; then shall the first and fifth taken together ( $AG$ ) be the same multiple of the second  $C$ , as the third and sixth taken together ( $DH$ ) is of the fourth  $F$ .

a 2. ax.

The number of parts in  $AB$  equal each to  $C$  is put equal to the number of parts in  $DE$ , whereof each part is equal to  $F$ . Likewise the number of parts  $BG$  is put equal to the number of parts in  $EH$ . Therefore the number of parts in  $AB + BG$  is equal to the number of parts in  $DE + EH$ . (a) That is, the whole line  $AG$  is the same multiple of  $C$ , as the whole line  $DH$  is of  $F$ . Which was to be demonstrated.

## P R O P. III. Fig. 7.

If the first  $A$  be the same multiple of the second  $B$ , as the third  $C$  is of the fourth  $D$ , and there be taken  $EI$ ,  $FM$  equimultiples of the first and third, then will each of the magnitudes taken be equimultiples, the one  $EI$  of the second  $B$ , the the other  $FM$  of the fourth  $D$ .

a hyp.

b 2. 5.

c 2. 5.

Let  $EG$ ,  $GH$ ,  $HI$ , the parts of the multiple  $EI$ , be equal to  $A$ ; also let  $FK$ ,  $KL$ ,  $LM$ , the parts of the multiple  $FM$  be equal to  $C$ ; (a) the number of these is equal to the number of those. Moreover  $A$  (that is)  $EG$  or  $GH$  or  $HI$ , is put the same multiple of  $B$ , as  $C$ , or  $FK$ , &c. is of  $D$ . (b) Therefore  $EG + GH$  is the same multiple of the second  $B$ , as  $FK + KL$  is of the fourth  $D$ . (c) By the same way of arguing is  $EI$  ( $EH + HI$ ) the same multiple of  $B$ , as  $FM$  ( $FL + LM$ ) is of  $D$ . Which was to be demonstrated.

## P R O P. IV. Fig. 8.

If the first  $A$  have the same ratio to the second  $B$ , as the third  $C$  to the fourth  $D$ ; then also  $E$  and  $F$  the equimultiples of the first  $A$  and the third  $C$ , shall have the same ratio to  $G$  and  $H$ , the equimultiples of the second  $B$  and the fourth  $D$ , according to any multiplication, if so taken as they answer each to other ( $E : G :: F : H$ .) Take



Take I and K equimultiples of E and F; and also L and M equimultiples of G and H. (a) Then is I the same multiple of A, as K is of C; (a) and also L is the same multiple of B, as M of D. Therefore whereas it is  $A : B :: C : D$ ; according to the sixth definition, if I be  $\square$ ,  $=$ ,  $\supset$  L, then consequently after the same manner is K  $\square$ ,  $=$ ,  $\supset$  M, Therefore when I and K are taken the same multiples of E and F, as L and M of G, and H, then will it be by the seventh definition  $E : G :: F : H$ . Which was to be demonstrated.

a 3. 5.

b hyp.

Coroll.

From hence is generally demonstrated the proof of inverse ratio. For because  $A : B :: C : D$ , therefore if E  $\square$ ,  $=$ ,  $\supset$  G, then is (c) likewise F  $\square$ ,  $=$ ,  $\supset$  H; therefore it is evident, that if G  $\square$ ,  $=$ ,  $\supset$  E, then is H  $\square$ ,  $=$ ,  $\supset$  F; (d) therefore  $B : A :: D : C$ . Which was to be demonstrated.

c 2. def. 5.

a 6. def. 5.

PROP. V. Plate III. Fig. 9.

If a magnitude AB be the same multiple of a magnitude CD, as a part taken from the one AE is of a part taken from the other CF; the residue of the one EB, shall be the same multiple of the residue of the other FD as the whole AB is of the whole CD.

Take another GA, which shall be the same multiple of FD, the residue, as AB is of the whole CD, or as the part taken away AE, is of the part taken away CF. (a) Therefore the whole GA + AE is the same multiple of the whole CF + FD, as the one AE is of the one CF, that is, as AB is of CD; therefore GE (b) = AB; and (c) so AE, which is common, being taken away, there remains GA = EB. Therefore, &c.

a 1. 5.

b 6. ax.

c 3. ax.

PROP. VI. Fig. 10.

If two magnitudes AB, CD, are equimultiples of two magnitudes E, F; and some magnitudes AG and CH equimultiples of the same E, F, be taken away; then the residues GB, HD, are either equal to these magnitudes E, F, or else equimultiples of them.

For because the number of parts in AB, whereof each is equal to E, is put equal to the number of parts in CD, whereof each is equal to F; and also the number of

E

parts

a 3. ax.

parts in AG equal to the number of parts in CH; If from one you take AG, and from the other CH, (a) there remains the number of parts in the remainder GB equal to the number of parts in HD; therefore if GB be once E, then is HB once C; if GB be many times E, then is HD so many times C. Which was to be demonstrated.

## PROP. VII. Plate III. Fig. 11.

Equal magnitudes A and B have the same proportion or ratio to the same magnitude C. And one the same magnitude C hath the same ratio to equal magnitudes A and B.

a 6. ax.

b 6. def. 5.

c cor. 4 5.

Take D and E equimultiples of the equal magnitudes A and B, and F any multiple of C; then is D (a) = E. Wherefore if D  $\square$ , =,  $\sqsupset$  F, then also E will be  $\square$ , =,  $\sqsupset$  F; (b) therefore A : C :: B : C; and (c) by inversion C : A :: C : B. Which was to be demonstrated.

Schol.

If instead of the multiple F, two equimultiples be taken, it will be the same way proved that equal magnitudes have the same ratio to any other magnitudes that are equal between themselves.

## PROP. VIII. Fig. 12.

Of unequal magnitudes AB, AC, the greater AB hath a greater ratio to the same third line D, than the lesser AC, and the same third line D hath a greater ratio to the lesser AC, than to a greater AB.

Take EF, EG equimultiples of the said AB, AC, so that EH, a multiple of D, may be greater than EG, but lesser than EF, (which will easily happen, if both EG and GF be taken greater than D.) It is manifest from

def. 5. that  $\frac{AB}{D} \square \frac{AC}{D}$  and  $\frac{D}{AB} \sqsupset \frac{D}{AC}$ .

Which was to be demonstrated.

## PROP. IX. Fig. 13.

Magnitudes, which to one and the same magnitude have the same ratio, are equal the one to the other. And if a magnitude have the same ratio to other magnitudes, those magnitudes are equal one to the other.

s. Hyp.

1. Hyp. If  $A : C :: B : C$ ; I say that  $A = B$ . For

Let  $A$  be greater or less than  $C$ ; (a) then is  $\frac{A}{C} > \frac{B}{C}$  or  $\frac{A}{C} < \frac{B}{C}$ . a 8. 5.

Which is contrary to the Hypothesis.

2. Hyp. If  $C : B :: C : A$ . I say that  $A = B$ . For let

be  $\frac{C}{B} > \frac{C}{A}$ , (b) then  $\frac{C}{B} > \frac{C}{A}$ . Which is against the Hypothesis. b 8. 5.

PROP. X. Plate III. Fig. 14.

Of magnitudes having ratio to the same magnitude, that which has the greater ratio, is the greater magnitude; and that magnitude to which the same bears a greater ratio, is the lesser magnitude.

1. Hyp. If  $\frac{A}{C} > \frac{B}{C}$ , I say that  $A > B$ .

For if it be said that  $A = B$ , (a) then  $A : C :: B : C$ . Which is contrary to the Hypothesis. If  $A < B$ , (b) then is

$\frac{A}{C} < \frac{B}{C}$  Which is also against the Hypothesis.

2. Hyp. If  $\frac{C}{B} > \frac{C}{A}$ , I say that  $B < A$ ; for if

we say  $B = A$ , it's against the Hypothesis, for it will follow that  $C : B :: C : A$ . If you say  $B > A$ , (d) then

$\frac{C}{B} < \frac{C}{A}$ . Which is also against the Hypothesis.

PROP. XI. Fig. 15, 16, 17.

Proportions which are one and the same to any third, are the same one to another.

Let  $A : B :: E : F$ , and  $C : D :: E : F$ . I say that  $A : B :: C : D$ . Take  $G, H, I$ , equimultiples of  $A, C, E$ ;

and  $K, L, M$ , equimultiples of  $B, D, F$ . Now (a) be-

cause  $A : B :: E : F$ ; if  $G < K$ , (b) then after

the same manner  $I < M$ . And likewise (a)

because  $E : F :: C : D$ , if  $I < M$ , (b) then is

likewise  $G < K$  (c) wherefore  $A : B :: C : D$ .

which was to be demonstrated.



Schol.

Proportions that are one and the same to the same proportions, are the same betwixt themselves.

P R O P. XII. Plate III. Fig. 15, 16, 17.

If any number of magnitudes A, B, C, D, E and F be proportional; as one of the antecedents A, is to one of the consequents B, so are all the antecedents A, C, E, to all the consequents, B, D, F.

a 1. 5.

b hyp.

c 6. def. 5.

Take G, H, I, equimultiples of the antecedents, and K, L, M, of the consequents. Because that one G is the same multiple of one A, (a) as all G, H, I, are of all A, C, E; and because one K is the same multiple of one B, as all K, L, M, are of all B, D, F: Moreover because  $A : B (b) :: C : D (b) :: E : F$ , if G be  $\square$ , or  $\sqsupset$  K, then will H likewise be  $\square$ , or  $\sqsupset$  L, and I  $\square$ , or  $\sqsupset$  M; and so if G  $\square$ , or  $\sqsupset$  K, in like manner will  $G + H + I$  be  $\square$ , or  $\sqsupset$   $K + L + M$  (c) wherefore  $A : B :: A + C + E : B + D + F$ . Which was to be demonstrated.

Corol.

Hence, if like proportionals be added to like proportionals, the wholes shall be proportional.

P R O P. XIII. Fig. 15, 16, 17.

If the first A have the same ratio to the second B, as the third C hath to the fourth D, and if the third C have a greater proportion to the fourth D, than the fifth E to the sixth F; then also shall the first A have a greater proportion to the second B, than the fifth E to the sixth F.

a 6. def. 5.

b 8. def. 5.

c 8. def. 5.

Take G, H, I, equimultiples of A, C, E, and K, L, M, equimultiples of B, D, F. Now because that  $A : B :: C : D$ , if H  $\square$  L, (a) then is G  $\square$  K; but because

$$\frac{C}{D} < \frac{E}{F},$$

(b) it may be that H  $\square$  L, and yet I not  $\square$  M.

$$\frac{A}{B} < \frac{E}{F}.$$

(c) Therefore  $\frac{A}{B} < \frac{E}{F}$ . Which was to be demonstrated.

Schol.

But if  $\frac{C}{D} \supset \frac{E}{F}$ , then also is  $\frac{A}{B} \supset \frac{E}{F}$ . Also, if

$\frac{C}{D} \sqsubset \frac{E}{F}$ , then is  $\frac{A}{B} \sqsubset \frac{E}{F}$ . And if  $\frac{A}{B} \supset$

$\frac{C}{D} \supset \frac{E}{F}$ , then is  $\frac{A}{B} \supset \frac{E}{F}$ .

PROP. XIV. Plate III. Fig. 18.

If the first A have the same ratio to the second B, that the third C hath to the fourth D; and if the first A be greater than the third C; then shall the second B be greater than the fourth D. But if the first A be equal to the third C, then the second B shall be equal to the fourth D; but if A be less, then is B also less.

Let  $A \sqsubset C$ ; (a) then  $\frac{A}{B} \sqsubset \frac{C}{B}$ , (b) but  $\frac{A}{B} = \frac{C}{D}$ ;

a 8. 5.  
b hyp.

(c) therefore  $\frac{C}{D} \sqsubset \frac{C}{B}$ ; (c) therefore  $B \sqsubset D$ . By the

c 13. 5.

like way of argument, if  $A \supset C$ , (d) then is  $B \supset D$ .

d 10. 5.

But if A be put equal to C, then  $C : B :: (e) A : B$  (f)

e 7. 5.

$:: C : D$ . (g) Therefore  $B = D$ . Which was to be demonstrated.

f hyp.

g 11. 5. &

9. 5.

Schol.

By an argument à fortiori, if  $\frac{A}{B} \supset \frac{C}{D}$ , and  $A \sqsubset C$ ,

then is  $B \sqsubset D$ . Likewise if  $A = B$ , then is  $C = D$ , and if  $A \sqsubset$ , or  $\supset B$ , then also is  $C \sqsubset$  or  $\supset D$ .

PROP. XV. Fig. 19.

Parts C and F are in the same ratio, with their like multiples AB and DE, if taken correspondently. ( $AB : DE :: C : F$ .)

Let AG, GB, parts of the multiple AB, be equal to C; and let DH, HE, parts of the multiple DE, be equal to F. (a) The number of these parts is equal to the num-

a hyp.

E 3

ber

b 7. 5.  
c 12. 5.

ber of those. Therefore whereas (b)  $AG : C :: DH : F$ , and  $GB : C :: HE : F$ ; therefore is (c)  $AG + GB (AB) : DH + HE (DE) :: C : F$ . Which was to be demonstrated.

P R O P. XVI. Plate III. Fig. 20.

If four magnitudes A, B, C, D, are proportional, they also shall be alternately proportional ( $A : C :: B : D$ )

a 15. 5.  
b hyp.  
c 11. 5. &  
l 14. 5.  
d 6. def. 5.

Take E and F equimultiples of A and B; take also G and H equimultiples of C and D. Therefore  $E : F (a) :: A : B (b) :: C : D (a) :: G : H$ . Wherefore if  $E \sqsubset, =, \supset (c) G$ , then likewise is  $F \sqsubset, =, \supset H (d)$ . Therefore  $A : C :: B : D$ . Which was to be demonstrated.

Schol.

Alternate ratio has place only when the quantities are of the same kind. For heterogeneous quantities are not compared together.

P R O P. XVII. Fig. 21.

If magnitudes compounded are proportional ( $AB : CB :: DF : FE$ .) they shall be proportional also when divided. ( $AC : CB :: DF : FE$ .)

a 1. 5.  
b constr.  
c 1. 5.  
d 2. 5.  
e 6. def. 5.  
f 5. ax.  
g 6. def. 5.

Take GH, HL, IK, KM, in order equimultiples of AC, CB, DF, FE; and also LN, MO, equimultiples of CB, FE. The whole GL is (a) the same multiple of the whole AB, as one GH is of one AC; (b) that is, as IK of DF, (c) or as the whole IM of the whole DE. Also HN (HL + LN) is (d) the same multiple of CB, as KO, (KM + MO) is of FE. Therefore, whereas by Hyp.  $AB : BC :: DE : EF$ , if GL be  $\sqsubset, =, \supset$  HN, then likewise (e) will IM  $\sqsubset, =, \supset$  KO. Take from these HL, KM, that are equal; and if the remainder GH be  $\sqsubset, =, \supset$  LN, (f) then will IK  $\sqsubset, =, \supset$  MO; (g) whence  $AC : CB :: DF : FE$ . Which was to be demonstrated.

P R O P. XVIII. Fig. 22.

If magnitudes divided are proportional ( $AB : BC :: DE : EF$ .) the same also being compounded shall be proportional ( $AC : CB :: DF : FE$ .)

a 17. 5.  
b hyp. &  
11. 5.  
c 14. 5.  
d 9. ax.

For if it can be, let  $AC : CB :: DF : FG \supset FE (a)$ . Then by division will  $AB : BC :: DG : GF$ ; (b) that is,  $DG : GF :: DE : EF$ ; and since DG  $\sqsubset$  DE, (c) therefore is GF  $\sqsubset$  EF. Which is absurd. The like absurdity will follow if it be said  $AC : CB :: DF : GF \sqsubset FE (d)$ .

P R O P.



PROP. XIX. Plate III, Fig. 23.

If the whole AB be to the whole DE as the part taken away AC is to the part taken away DF, then shall the residue CB be to the residue FE, as the whole AB is to the whole DE.

Because (a)  $AB : DE :: AC : DF$  (b) therefore by permutation  $AB : AC :: DE : DF$ , (c) and thence by division  $AC : CB :: DF : FE$ ; (b) wherefore again by permutation  $AC : DF :: CB : FE$  (d) that is,  $AB : DE :: CB : FE$ . Which was to be demonstrated.

Coroll.

Hence, If like proportionals be subtracted from like proportionals, the remainders shall be proportional.

2. Hence is converse ratio demonstrated.

Let  $AB : CB :: DE : FE$ . I say that  $AB : AC :: DE : DF$ . For by (a) permutation  $AB : DE :: CB : FE$ , (b) therefore  $AB : DE :: AC : DF$ , whence again by permutation  $AB : AC :: DE : DF$ . Which was to be demonstrated.

PROP. XX. Fig. 24.

If there are three magnitudes A, B, C, and others D, E, F, equal to those in number, which being taken two and two in each order are in the same ratio, ( $A : B :: D : E$ ; and  $B : C :: E : F$ ;) and if ex æquo the first A be greater than the third C; then shall the fourth D be greater than the sixth F. But if the first A be equal to the third C, then the fourth D shall be equal to the sixth F; and if A be less than C, then shall D be less than F.

1. Hyp. If  $A < C$ . Because (a)  $E : F :: B : C$ , by (b) inversion it shall be  $F : E :: C : B$ . (c) But  $\frac{C}{B} > \frac{A}{B}$ , (d) there-

fore  $\frac{F}{E} > \frac{A}{B}$  or  $\frac{D}{E}$ , (e) therefore  $D > F$ . Which was to be demonstrated.

2. Hyp. By the same way of arguing, if  $A = C$ , it will appear that  $D = F$ .

3. Hyp. If  $A > C$ . Because  $F : E :: C : B :: (f) A : B :: D : E$ , (g) therefore is  $D = F$ . Which was to be dem.

PROP. XXI. Fig. 24.

If there are three magnitudes A, B, C, and others also D, E, F, equal to them in number, which taken two and two

E 4

are

a hyp.  
b 16. 5.  
c 17. 5.  
d hyp. &  
11. 5.

a 16. 5.  
b 19. 5.

a hyp.  
b cor. 4. 5.  
c hyp. and  
8. 5.  
d schol. 13. 5  
e 10. 5.

f 7. 5.  
g 11. 5. &  
9. 5.

are in the same ratio; and their proportion perturbate ( $A : B :: E : F$ , and  $B : C :: D : E$ .) and if ex æquo the first  $A$  be greater than the third  $C$ , then is the fourth  $D$  greater than the sixth  $F$ ; but if the first be equal to the third, then is the fourth equal to the sixth; if less, so is the other likewise.

a hyp.

b 8. 5.

c sch. 13. 5.

d 10. 5.

1. Hyp. If  $A \sqsubset C$ ; then because (a)  $D : E :: B : C$ ,  
therefore inversely  $E : D :: C : B$ , but  $\frac{C}{B} (b) \supset \frac{A}{B}$ ;

(c) therefore  $\frac{E}{D} \supset \frac{A}{B}$ , that is, than  $\frac{E}{F}$ , (d) therefore

$D \sqsubset F$ .

2. Hyp. By the like argument, if  $A \supset C$ , then is  $D \supset F$ .

e 7. 5.

f hyp.

g 9. 5.

3. Hyp. If  $A = C$ ; then because  $E : D :: (e) C : B$   
 $:: (e) A : B :: (f) E : F$ , (g) therefore is  $D = F$ .  
Which was to be demonstrated.

PROP. XXII. Plate III. Fig. 23.

If there be any number of magnitudes  $A, B, C$ , and others equal to them in number  $D, E, F$ , which taken two and two are in the same ratio ( $A : B :: D : E$  and  $B : C :: E : F$ .) they shall be in the same ratio also by equality ( $A : C :: D : F$ .)

Take  $G, H$ , equimultiples of  $A, D$ ; and  $I, K$ , of  $B, E$ ; and also  $L, M$ , of  $E, F$ .

a hyp.

b 4. 5.

c 20. 5.

d 6. def. 5.

Because (a)  $A : B :: D : E$ , (b) therefore  $G : I :: H : K$ ; and in like manner  $I : L :: K : M$ . therefore if  $G \sqsubset, =, \supset L$ , (c) then is  $H \sqsubset, =, \supset M$ ; (d) therefore  $A : C :: D : F$ . By the same way of demonstration if further  $C : N :: F : O$ , then by equality  $A : N :: D : O$ . Which was to be demonstrated.

PROP. XXIII. Fig. 26.

If there are three magnitudes  $A, B, C$ , and others  $D, E, F$ , equal to them in number, which taken two and two are in the same ratio, and their proportion perturbate ( $A : B :: E : F$ , and  $B : C :: D : E$ .) they shall be in the same ratio also by equality ( $A : C :: D : F$ .)

Take  $G, H, I$ , equimultiples of  $A, B, D$ ; and also  $K, L, M$ , equimultiples of  $C, E, F$ . Then  $G : H :: (a) A : B :: (b) E : F (a) :: L : M$ . Moreover because (b)  $B : C :: D : E$ , thence is (c)  $H : K :: I : L$ ; therefore  $G, H, K$ , and  $I, L, M$ , are as in 21. 5. Therefore if

a 15. 5.

b hyp.

c 4. 5.

G be  $\sqsubset$ ,  $=$ ,  $\sqsupset$  K, then is likewise I  $\sqsubset$ ,  $=$ ,  $\sqsupset$  M, and so (d) consequently  $A : C :: D : F$ . Which was to be demonstrated.

d 6. def. 5.

If there are more magnitudes than three, this way of demonstration holds good in them also.

Coroll.

From hence \* it follows, that ratio's compounded of the same ratio's, are among themselves the same; as also that the same parts of the same ratio's, are among themselves the same.

\* 22<sup>or</sup> 23.  
5. and 20.  
def. 1.

P R O P. XXIV. Plate III. Fig. 27.

If the first magnitude AB, has the same ratio to the second C, which the third DE, has to the fourth F; and if the fifth BG has the same ratio to the second C, which the sixth EH has to the fourth F; then shall the first compounded with the fifth (AG) have the same ratio to the second C, which the third compounded with the sixth (DH) has to the fourth F.

For because (a)  $AB : C :: DE : F$ , and by the Hyp. and inversion  $C : BG :: F : EH$ ; therefore by (b) equality  $AB : BG :: DE : EH$ , whence by compounding,  $AG : BG :: DH : EH$ . Also (c)  $BG : C :: EH : F$ . Therefore again by (b) equality  $AG : C :: DH : F$ . Which was to be demonstrated.

a hyp.  
b 22, 5.  
c hyp.

P R O P. XXV. Fig. 28.

If four magnitudes are proportional ( $AB : CD :: E : F$ ) the greatest AB and the least F shall be greater than the rest C, D, and E.

Make  $AG = E$ , and  $CH = F$ . Because  $AB : CD ::$  (a)  $E : F ::$  (b)  $AG : CH$ , (c) thence is  $AB : CD :: GB : HD$ ; (d) but  $AB \sqsubset CD$ , (e) therefore  $GB \sqsubset HD$ . But  $AG + F = E + CH$ , therefore  $AG + F + GB \sqsubset E + CH + HD$ ; that is,  $AB + F \sqsubset E + CD$ . Which was to be demonstrated.

a hyp.  
b 7. 5.  
c 19. 5.  
d hyp.  
e sch. 14. 5.

These propositions which follow are not Euclide's, but taken out of other Authors, and are here subjoyned because of their frequent use.

P R O P. XXVI. Fig. 29.

If the first have a greater proportion to the second, than the third to the fourth; then by conversion, the second shall have a less proportion to the first, than the fourth to the third.

Let



Let  $\frac{A}{B} \sqsubset \frac{C}{D}$ , I say that  $\frac{B}{A} \sqsupset \frac{D}{C}$ . For conceive  
 $\frac{C}{D} = \frac{E}{B}$ ; (a) therefore  $\frac{A}{B} \sqsubset \frac{E}{B}$ ; (b) whence  $A \sqsubset E$ ; (c) there-  
 fore  $\frac{B}{A} \sqsupset \frac{B}{E}$ , (d) or  $\frac{B}{A} \sqsupset \frac{D}{C}$ . Which was so be demonstrated.

a 13. 5.  
 b 10. 5.  
 c 8. 5.  
 d cor. 4. 5.

## PROP. XXVII. Plate III. Fig. 29.

If the first have a greater proportion to the second, than the third to the fourth; then alternately the first shall have a greater proportion to the third, than the second to the fourth.

Let  $\frac{A}{B} \sqsubset \frac{C}{D}$ , then I say  $\frac{A}{C} \sqsubset \frac{B}{D}$ . For conceive  
 $\frac{C}{D} = \frac{E}{B}$ , (a) therefore  $A \sqsubset E$ , (b) and therefore  $\frac{A}{C} \sqsubset \frac{E}{B}$ ,  
 (c) or  $\frac{A}{C} \sqsubset \frac{B}{D}$ . Which was to be demonstrated.

a 10. 5.  
 b 1. 5.  
 c 16. 5.

## PROP. XXVIII. Fig. 30.

If the first have a greater proportion to the second than the third to the fourth, then the first compounded with the second shall have a greater proportion to the second, than the third compounded with the fourth, to the fourth.

Let  $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$ . I say that  $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$ . For  
 conceive  $\frac{GB}{BC} = \frac{DE}{EF}$ , (a) therefore is,  $AB \sqsubset GB$ , add  
 BC to each part, then (b) will  $AC \sqsubset GC$ ; (c) therefore  
 $\frac{AC}{BC} \sqsubset \frac{GC}{BC}$ , (d) that is,  $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$ . Which was to be dem.

a 10. 5.  
 b 4. an.  
 c 8. 5.  
 d 18. 5.

## PROP. XXIX. Fig. 30.

If the first compounded with the second has a greater proportion to the second, than the third compounded with the

the fourth hath to the fourth; then by division the first shall have a greater proportion to the second, than third to the fourth.

Let  $\frac{AC}{BC} \supset \frac{DF}{EF}$ , then I say  $\frac{AB}{BC} \supset \frac{DE}{EF}$ . For

conceive  $\frac{AC}{BC} = \frac{GC}{EF}$ , (a) therefore  $AC \supset GC$ . Take a 10. 5.

away BC, which is common, there (b) remains  $AB \supset$  b 5. and a

$GB$ ; (c) therefore  $\frac{AB}{BC} \supset \frac{GB}{BC}$  (d) or  $\frac{DE}{EF}$ . Which c 8. 4. d

was to be demonstrated. d 17. 5.

PROP. XXX. Plate III. Fig. 31.

If the first compounded with the second, has a greater proportion to the second, than the third compounded with the fourth, hath to the fourth; then by converse ratio shall the first compounded with the second have a lesser ratio to the first, than the third compounded with the fourth shall have to the third.

Let  $\frac{AC}{BC} \supset \frac{DF}{EF}$ . Then I say that  $\frac{AC}{AB} \supset \frac{DF}{DE}$

For because that  $\frac{AC}{BC} \supset \frac{DF}{EF}$ , (a) therefore, by division, a hyp. b 29. 5.

$\frac{AB}{BC} \supset \frac{DE}{EF}$ , by conversion (c) therefore  $\frac{BC}{AB} \supset \frac{EF}{DE}$  c 26. 5.

and (d) therefore by composition  $\frac{AC}{AB} \supset \frac{DF}{DE}$ . Which d 28. 5.

was to be demonstrated.

PROP. XXXI. Fig. 32.

If there are three magnitudes A, B, C, and others also D, E, F, equal to them in number; and if there be a greater proportion of the first of the former to the second, than there is of the first of the last to the second  $\left(\frac{A}{B} \supset \frac{D}{E}\right)$  and there be also a greater proportion of the second of the first magnitudes to their third, than there is

is of the second of the last magnitudes to their third  
 $\left(\frac{B}{C} \sqsubset \frac{E}{F}\right)$  Then by equality also shall the ratio of the  
 first of the former magnitudes to the third, be greater than  
 the ratio of the first of the latter magnitudes, to the third  
 $\left(\frac{A}{C} \sqsubset \frac{D}{F}\right)$

a 10. 5.

b 8. 5.

c 13. 5.

d 10. 5.

e 8. 2.

f 32. 5.

Conceive  $\frac{G}{C} = \frac{E}{F}$ , (a) then is  $B \sqsubset G$ , and (b) there-

fore  $\frac{A}{G} \sqsubset \frac{A}{B}$ . Again conceive  $\frac{H}{G} = \frac{D}{E}$ , (c) then

$\frac{H}{G} \supset \frac{A}{B}$ , therefore much more  $\frac{H}{G} \supset \frac{A}{G}$ , (d) wherefore

$A \sqsubset H$ , (e) and consequently  $\frac{A}{C} \sqsubset \frac{H}{C}$ , (f) or  $\frac{D}{F}$ . W.W.D.

PROP. XXXII. Plate III. Fig. 32.

If there be three magnitudes A, B, C, and others  
 D, E, F, equal to them in number; and there be a  
 greater proportion of the first of the former magnitudes to  
 the second, than there is of the second of the latter to  
 the third  $\left(\frac{A}{B} \sqsubset \frac{E}{F}\right)$  and also the ratio of the se-  
 cond of the former to the third be greater than the ratio  
 of the first of the latter to the second  $\left(\frac{B}{C} \sqsubset \frac{D}{E}\right)$   
 then by equality also shall the proportion of the first of  
 the former to the third, be greater than that of the first  
 of the latter to the third  $\left(\frac{A}{C} \sqsubset \frac{D}{F}\right)$

a 10. 5.

b 8. 5.

c schol.

13. 5.

Suppose  $\frac{G}{C} = \frac{D}{E}$  then is (a)  $B \sqsubset G$ , and therefore

(b)  $\frac{A}{G} \sqsubset \frac{A}{B}$ . Again, Suppose  $\frac{H}{G} = \frac{E}{F}$ ; therefore is (c)

$\frac{H}{G} \supset \frac{A}{B}$ , and consequently (a)  $A \sqsubset H$ , and thence (b)



$\frac{A}{C} = \frac{H}{C}$  (d) or  $\frac{D}{F}$ . Which was to be demonstrated.

d 13. 5.

PROP. XXXIII. Plate III. Fig. 33.

If the proportion of the whole AB to the whole CD be greater than the proportion of the part taken away AE to the part taken away CF; then shall also the ratio of the remainder EB to the remainder FD be greater than that of the whole AB to the whole CD.

Because that  $\frac{AB}{CD}$  (a)  $>$   $\frac{AE}{CF}$  (b) therefore by permutation  $\frac{AB}{AE}$   $>$   $\frac{CD}{CF}$ , (c) therefore by converse ratio  $\frac{AB}{CD}$   $>$   $\frac{EB}{FD}$ , and by permutation again  $\frac{AB}{CD}$   $>$   $\frac{EB}{FD}$ .

a hyp.

b 27. 1.

c 30. 5.

Which was to be demonstrated.

PROP. XXXIV.

If there be any number of magnitudes, and others also equal to them in number; and the proportion of the first of the former to the first of the latter be greater than that of the second to the second, and that greater than the proportion of the third to the third, and so forward: all the former magnitudes together shall have a greater ratio to all the latter together, than all the former, leaving out the first, shall have to the latter, leaving out the first; but less than that of the first of the former to the first of the latter; and lastly, greater than that of the last of the former to the last of the latter.

You may please to consult Interpreters for the demonstration hereof, we having for brevity sake omitted it, and because 'tis of no use in these Elements.

The End of the fifth Book.

The

The SIXTH BOOK

O F

E U C L I D E S  
E L E M E N T S.

Definitions.

Platē III.  
Fig. 33.

I. Like right-lined figures (ABC, DCE) are such whose several angles are equal one to the other, and also their sides about the equal angles, proportional.

*The Angle B = DCE, and AB : BC :: DC : CE.  
Also the angle A = D, and BA : AC :: CD : DE. Lastly the angle ACB = E, and BC : CA :: CE : ED.*

Fig. 34.

II. Reciprocal figures are (BD, BF) when in each of the figures there are terms both antecedent and consequent (that is, AB : BG :: EB : BC).

Fig. 23.

III. A right-line AB is said to be cut according to extreme and mean proportion, when as the whole AB is to the greater segment AC, so is the greater segment AC to the less CB (AB : AC :: AC : CB.)

Fig. 33.

IV. The altitude of any figure ABC, is a perpendicular line AG, drawn from the top A, to the base BC.

V. A ratio is said to be compounded of ratio's, when the quantities of the ratio's, being multiplied into one another, produce a ratio, As the ratio of A to C is

compounded of the ratio's of A to B and B to C. For  $\frac{A}{B} \times \frac{B}{C} = \frac{A}{C}$

a 20. def. 5. (a) =  $\frac{AB}{BC}$  (b) =  $\frac{A}{C}$ .  
b 15. 5.

PROP. I. Plate III. Fig. 35

Triangles ABC, ACD, and parallelograms BCAE, CDFA, which have the same height, are to each other, as their bases, BC, CD.

1

(a) Take

(a) Take as many as you please, BG, GH, equal to BC, and also DI=CD, and join AG, AH, AL.  
 (b) The triangles ACB, ABG, AGH, are equal, and also the triangle ACD=ADI. Therefore the triangle ACH is the same multiple of the triangle ACB, as the base HC is of the base BC; and the triangle ACI the same multiple of the triangle ACD, as the base CI is of CD. But if  $HC = CI$ , (c) then is likewise the triangle AHC = ACI; and (d) therefore  $BC : CD ::$  the triangle ABC : ACD :: (e) Pgr. CE : CF. Which was to be demonstrated.

a 3. 1.

b 38. 1.

c sch. 38. 1.

d 6. def. 5.

e 41. 1. 5.

15. 5.

Schol. Plate III. Fig. 37.

Hence triangles, ABC, DEF, and Pgrs. AGBC, DEFH, whose bases BC, EF are equal, are to each other as their altitudes, AI, DK.

(a) Take  $IL = CB$ , and  $KM = EF$ ; and join LA, G, MD, MH, then it is evident, that the triangle ABC : DEF :: (b) ALI : DKM :: (c) AI : DK :: (d) Pgr. AGBC : DEFH. Which was to be demonstrated.

a 3. 1.

b 7. 5.

c 1. 6.

d 41. 1. 5.

15. 5.

P R O P. II. Fig. 36.

If to one side BC of a triangle ABC, be drawn a parallel right-line DE, the same shall cut the sides of the triangle proportionally ( $AB : BD :: AE : EC$ .) And if the sides of the triangle are cut proportionally ( $AD : DB :: AE : EC$ ) then a right-line DE, joining the points of a section DE, shall be parallel to BC, the other side of the triangle. Draw CD and BE.

1. Hyp. Because the triangle DEB (a) = DEC, (b) therefore shall be the triangle ADE : DBE :: ADE : ECD. But the triangle AED : DBE :: (c) AD : DB, and the triangle ADE : DEC :: AE : EC; (d) therefore AD : DB :: AE : EC.

2. Hyp. Because  $AD : DB :: AE : EC$ , (e) that is as the triangle ADE : DBE :: ADE : ECD; (f) therefore the triangle DBE = ECD; and (g) therefore DE, BC are parallels. Which was to be demonstrated.

a 37. 1.

b 7. 5.

c 1. 6.

d 11. 5.

e 1. 6.

f 9. 5.

g 39. 1.

Schol.

If there are drawn several lines DE, FG parallel to the side BC of a triangle, all the segments of the sides shall be proportional.

For



a 2. 6.

For  $DF : FA (a) :: EG : GA$ ; and compounding and inverting,  $FA : DA :: GA : EA$ ; (a) and  $DA : DB :: EA : EC$ ; therefore by equality  $DF : DB :: EG : EC$ . Which was to be demonstrated.

Coroll.

If  $DF : DB :: EG : EC$ ; (a) then  $BC, DE, FG$  shall be parallels.

## PROP. III. Plate III. Fig. 38.

If an angle  $BAC$  of a triangle  $BCA$  be bisected, and the right-line  $AD$ , that bisects the angle, cut the base also; then shall the segments of the base have the same ratio that the other sides of the triangle have, ( $BD : DC :: AB : AC$ ). And if the segments of the base have the same ratio that the other sides of the triangle have ( $BD : DC :: AB : AC$ ) then a right-line  $AD$  drawn from the top  $A$  to the section  $D$ , shall bisect that angle  $BAC$  of the triangle.

Produce  $BA$ , and make  $AE = AC$ , and join  $CE$ .

a 5. 1.

b 32. 1.

c hyp.

d 27. 1.

e 2. 6.

f 2. 6.

g 29. 1.

h 5. 1

k 1. ax.

1. Hyp. Because  $AE = AC$ , therefore is the angle  $ACE (a) = E (b) = \text{half } BAC (c) = DAC$ ; (d) therefore  $DA, CE$  are parallels. (e) Wherefore  $BA : AE (AC) :: BD : DC$ ,

2. Hyp. Because  $BA : AC (AE) :: BD : DC$ , (f) therefore are  $DA, CE$  parallels; and (g) therefore is the angle  $BAD = E$ ; and the angle  $DAC (g) = ACE (b) = E$ , (h) therefore the angle  $BAD = DAC$ . Wherefore the angle  $BAC$  is bisected. Which was to be demonstrated.

## PROP. IV. Fig. 39.

Of equiangular triangles  $ABC, DCE$ , the sides are proportional which are about the equal angles,  $B, DCE$ , ( $AB : BC :: DC : CE$ , &c.) And the sides  $AB, DC$ , &c. which are subtended under the equal angles  $ACB, E$ , &c. are homologous, or of like ratio.

a 32. 1. &amp;

13. ax.

b hyp.

c 28. 1.

d 34. 1.

e 2. 6.

Set the side  $EC$  in a direct line to the side  $CE$ , and produce  $BA$  and  $ED$  till they (a) meet in  $F$ .

Because the angle  $B (b) = ECD$ , (c) therefore  $BF, CD$  are parallel: Also because the angle  $BCA (b) = CED$ , (e) therefore are  $CA, EF$  parallel. Therefore the figure  $CAFD$  is a Pgr. (d) therefore  $AF = CD$ , and  $AC = FD$ . Whence it is evident, that  $AB : AF (CD) :: BC : FD$ .

BC : CE. (f) By permutation therefore AB : BC : : CD : CE, also BC : CE : : FD (AC) : DE. (f) And thence by permutation BC : AC : : CE : DE. (g) Wherefore also by equality AB : AC : : CD : DE. Therefore, &c.

f 16. 5.  
g 22. 5.

Coroll.

Hence AB : DC : : BC : CE : : AC : DE.

Schol.

Hence, if in a triangle FBE there be drawn AC, a parallel to one side FE, the triangle ABC shall be similar or like to the whole FBE.

PROP. V. Plate III. Fig. 40, 41.

If two triangles ABC, DEF, have their sides proportional (AB : BC : : DE : EF, and AC : BC : : DF : EF, and also AB : AC : : DE : DF.) those triangles are equiangular, and those angles equal, under which are subtended the homologous sides.

At the side EF (a) make the angle FEG = B, and the angle EFG = C; (b) whence the angle G = A. Therefore GE : EF (c) : : AB : BC : : (d) DE : EF. (e) And therefore GE = DE. Likewise GF : FE (e) : : AC : CE : : (d) DF : FE; (e) therefore GF = DF. Therefore the triangles DEF, GEF, are mutually equilateral. (f) Therefore the angle D = G = A, and the angle FED (f) = FEG = B, and (g) consequently the angle DFE = C. Therefore, &c.

a 23. 1.  
b 32. 1.  
c 4. 6.  
d hyp.  
e 11. 5. &c  
f 9. 5.  
f 8. 1.  
g 32. 1.

PROP. VI. Fig. 40, 41.

If two triangles ABC, DEF have one angle B equal to one angle D E F, and the sides about the equal angles B, DEF proportional (AB : BC : : DE : EF) then those triangles ABC, DEF, are equiangular, and have those angles equal, under which are subtended the homologous sides.

At the side EF make the angle FEG = B, and the angle EFG = C; (a) then will the angle G = A. Therefore GE : EF : : (b) AB : BC : : (c) DE : EF, (d) and therefore DE = GE. But the angle DEF (e) = B (f) = GEF; therefore the angle D (g) = G = A, (b) and consequently the angle EFD = C. Which was to be demonstrated.

a 32. 1.  
b 4. 6.  
c hyp.  
d 9. 5.  
e hyp.  
f constr.  
g 4. 1.  
h 32. 1.

F

PROP.

## P R O P. VII. Plate III. Fig. 43. 44.

If two triangles ABC, DEF have one angle A equal to one angle D, and the sides about the other angles ABC, DEF proportional ( $AB : BC :: DE : EF$ ) and if they have the remaining angles C, F, either both less or both greater than a right-angle; then shall the triangles ABC, DEF, be equiangular, and have those angles equal about which the proportional sides are.

a hyp.

b 32. 1.

c 4. 6.

d hyp.

e 9. 5.

f 5. 1.

g cor. 17. 1.

h cor. 13. 1.

For, if it can be, let the angle  $ABC = E$ , and make the angle  $ABG = E$ . Therefore, whereas the angle  $A (a) = D, (b)$  thence is the angle  $AGB = F$ . Therefore  $AB : BG (c) :: DE : EF :: (d) AB : BC, (e)$  therefore  $BG = BC; (f)$  therefore the angle  $BGC = BCG. (g)$  Therefore  $BGC$ , or  $C$ , is less than a right angle, and  $(b)$  consequently  $AGB$  or  $F$  is greater than right: Therefore the angles  $C$  and  $F$  are not of the same species or kind, which is against the Hypothesis.

## P R O P. VIII. Fig. 42.

If a line AD be drawn from the right-angle A, of a right-angled triangle ABC, perpendicular to the base BC, then the triangles ADB, ADC, on each side the perpendicular, are similar both to the whole ABC, and to another.

a hyp.

b 12. ax.

c 32. 1. &amp;

4. 6.

d vid. 21. 6. Which was to be demonstrated.

For because BAC, ADB are  $(a)$  right-angles,  $(b)$  and so equal, and B common; the triangles BAC, ADB, are like. By the same way of arguing BAC, ADC are like;  $(d)$  whence also ADB, ADC will be like.

Coroll.

e 1. def. 6.

Hence, 1.  $BD : DA (e) :: DA : DC.$   
2.  $BC : AC :: AC : DC$ , and  $CB : BA :: BA : BD$

## P R O P. IX. Fig. 45.

From a right-line given AB to cut off any part required as one third (AG.)

a 3. 1.

b 31. 1.

From the point A draw an infinite line AC any way in which  $(a)$  take any three equal parts AD, DE, EF, join FB, to which, from D,  $(b)$  draw the parallel DG, and the thing is done.



For  $GB : AG :: (c) FD : AD$ ; whence, by  $(d)$  composition,  $AB : AG :: AF : AD$ ; therefore since  $AD =$  one third of  $AF$ ,  $AG$  shall be  $=$  one third of  $AB$ . Which was to be done.

c 2. 6.  
d 18. 5.

PROP. X. Plate III. Fig. 46.

To divide a given undivided right-line,  $AB$  (in  $F, G$ ) as another given right-line is divided (in  $D, E$ .)

Let a right-line  $BC$  join the extremities of the line divided, and of the line not divided; and parallel to this, from the points  $E, D$ ,  $(a)$  draw  $EG, DF$ , meeting with the right-line which is to be cut in  $G$  and  $F$ ; then the thing is done.

a 31. 1.

For let  $DH$  be  $(a)$  drawn parallel to  $AB$ . Then  $AD : DE :: (b) AF : FG$ , and  $DE : EC (b) :: DI : IH :: (c) FG : GB$ . Which was to be done.

b 2. 6.  
c 34. 1. &  
7. 5.

Schol. Fig. 47.

Hence we learn to cut a right-line given  $AB$ , into as many equal parts as we please (suppose 5;) which will be more easily performed thus.

Draw an infinite line  $AD$ , and another  $BH$  parallel to it, and infinite also. Of these take equal parts,  $AR, RS, SV, VN$ ; and  $BZ, ZX, XT, TL$ ; in each line less parts by one, then are required in  $AB$ ; then let the right-lines  $LR, TS, XV, ZN$ , be drawn; these lines so drawn shall cut the right-line given  $AB$  into five equal parts.

For  $RL, ST, VX, NZ$ , are  $(a)$  parallels; therefore whereas  $AR, RS, SV, VN$  are  $(b)$  equal;  $(c)$  thence  $AM, MO, OP, PQ$ , are equal also. Likewise, because that  $BZ = ZX$ , therefore is  $BQ = PQ$ , and therefore  $AB$  is cut into five equal parts. Which was to be done.

a 33. 1.  
b constr.  
c 2. 6.

PROP. XI. Fig. 48.

Two right-lines being given  $AB, AD$ , to find out a third in proportion to them  $(DE)$ .

Join  $BD$ , and from  $AB$ , being produced, take  $BC = AD$ . Through  $C$  draw  $CE$  parallel to  $BD$ ; with which let  $AD$  produced meet in  $E$ , then is  $DE$  the proportional required.

For  $AB : BC (AD) (a) :: AD : DE$ . Which was to be done.

a 2. 6.

F 2

Or

Or thus: make the angle BDA (Fig. 42.) right, and also the angle BAC right, then (b)  $BD : DA :: DA : DC$ .

P R O P. XII. Plate III. Fig. 49.

Three right-lines being given, DE, EF, DG, to find a fourth proportional, GH.

a 2. 6.

Join EG, and through F draw FH parallel to EG with which let DG, produced to H, meet. Then it is evident that  $DE : EF (a) :: DG : GH$ . Which was to be done.

P R O P. XIII. Fig. 50.

Two right-lines being given AE, EB, to find a mean proportional, EF.

a 21. 3.

b cor. 8. 6.

Upon the whole line AB, as a diameter, describe a semicircle AFB, and from E erect a perpendicular EF meeting with the periphery in F, then  $AE : EF :: EF : EB$ . For let AF and FB be drawn; (a) then from the right-angle of the right-angled triangle AFB is drawn a right-line FE, perpendicular to the base. (b) Therefore  $AE : FE :: FE : EB$ . Which was to be done.

Coroll.

Hence, a right-line drawn in a circle from any point of a diameter, perpendicular to that diameter, and produced to the circumference, is a mean proportional betwixt the two segments of that diameter.

P R O P. XIV. Fig. 34.

Equal Parallelograms BD, BF, having one angle ABC equal to one EBG, have the sides which are about the equal angles reciprocal ( $AB : BG :: EB : BC$ ;) and those parallelograms BD, BF, which have one angle ABC equal to one EBG, and the sides which are about the equal angles reciprocal, are equal.

a sch. 15. 1.

b 1. 6.

c 7. 5.

d 1. 6.

e 11. 5.

For let the sides AB, BG, about the equal angles make one right-line; (a) wherefore EB, BC, shall do the same. Let FG, DC, be produced till they meet.

1. Hyp.  $AB : BG (b) :: BD : BH :: (c) BF : BH$  (d)  $BE : BC$ , (e) therefore, &c.

2. Hyp.

2. Hyp.  $BD : BH :: (f) AB : BG :: (g) BE : BC$   
 $:(h) BF : BH.$  (k) Therefore the Pgr.  $BD = BF.$   
*Which was to be demonstrated.*

f 1. 6.  
g hyp.  
h 1. 6.  
k 11. and  
9. 5.

PROP. XV. Plate III. Fig. 51.

Equal triangles having one angle  $ABC$ , equal to one  $DBE$ ,  
their sides which are about the equal angles are reciprocal  
 $AB : BE :: DB : BC.$  And those triangles that have  
one angle  $ABC$  equal to one  $DBE$ , and have also the sides  
that are about the equal angles reciprocal  $(AB : BE :: DB :$   
 $BC)$  are equal.

Let the sides  $CB, BD$ , which are about the equal an-  
gles be set in a strait-line; (a) therefore  $ABE$  is a right-  
line. Let  $CE$  be drawn.

1. Hyp.  $AB : BE :: (b)$  the triangle  $ABC : CBE$  (c)  
triangle  $DBE : CBE :: (d) DB : BC ; (e)$  therefore, &c.

2. Hyp. The triangle  $ABC : CBE :: (f) AB : BE ::$   
 $(g) DB : BC (h) ::$  the triangle  $DBE : CBE.$  (k) There-  
fore the triangle  $ABC = DBE.$  Which was to be demon-  
strated.

a sch. 15. 1.  
b 1. 6.  
c 7. 5.  
d 1. 6.  
e 11. 5.  
f 1. 6.  
g hyp.  
h 1. 6.  
k 11. and  
9. 5.

PROP. XVI. Fig. 52.

If four right-lines are proportional  $(AB : FG :: EF : CB)$   
the rectangle  $AC$ , comprehended under the extremes  $AB, CB$ ,  
is equal to the rectangle  $EG$ , comprehended under the means  
 $FG, EF.$  And if the rectangle  $AC$ , comprehended under the  
extremes  $AB, CB$ , be equal to the rectangle  $EG$ , compre-  
hended under the means  $FG, EF$ , then are the four right-  
lines proportional  $(AB : FG :: EF : CB.)$

1. Hyp. The angles  $B$  and  $F$  are right, and (a) conse-  
quently equal, and by hypothesis  $AB : FG :: EF : CB$ ,  
(b) therefore the rectangle  $AC = EG.$

2. Hyp. The rectangle  $AC$  (c)  $= EG$ , and the angle  
 $B = F ; (d)$  therefore  $AB : FG :: EF : CB.$  Which was  
to be demonstrated.

Coroll.

Hence it is easy to apply a rectangle given  $EG$  to a  
right-line given  $AB ; (viz.) (e)$  by making  $AB : EF ::$   
 $FG : BC.$

a 12. ax.  
b 14. 6.  
c hyp.  
d 14. 6.

e 12. 6.



## P R O P. XVII. Plate III. Fig. 52.

If three right-lines are proportional ( $AB : EF :: EF : CB$ ) the rectangle AC, made under the extremes AB, CB, is equal to the square EG, made of the middle EF. And if the rect. angle AC, comprehended under the extremes AB, CB, be equal to the square EG, made of the middle EF, then the three lines are proportional, ( $AB : EF :: EF : CB$ .)

Take  $FG = EF$ .

a hyp.

b 16. 6.

c 29. def. 1.

d hyp.

e 16. 6.

1. Hyp.  $AB : EF :: (a) EF (FG) : CB$ , therefore the rectangle AC (b) = EG (c) =  $EFq$ .

2. Hyp. The rectangle AC (d) = to the square EG =  $EFq$ ; (e) therefore  $AB : EF :: FG (EF) : BC$ . Which was to be demonstrated.

Coroll.

Let  $AXB = Cq$ , then  $A : C :: C : B$ .

## P R O P. XVIII. Fig. 53.

Upon a right-line given AB, to describe a right-lined figure AGHB, similar and alike situate to a right-lined figure given CEFD.

a 23. 1.

Resolve the right-lined figure given into triangles; (a) Make the angle  $ABH = D$ , [a] and the angle  $BAH = DCF$ , (a) and the angle  $AHG = CFE$ , (a) and the angle  $HAG = FCE$ , then AGHB, shall be the right-lined figure sought,

b constr.

c 32. 1.

For the angle B (b) = D, and the angle BAH (b) = DCF, (c) wherefore the angle AHB = CFD; (b) also the angle HAG = FCE, and the angle AHG (b) = CFE; (c) wherefore the angle G = E, and the whole angle GAB (d) = ECD, and the whole angle GAB (d) = EFD. The Polygons therefore are mutually equiangular. Moreover because the triangles are equiangular, therefore  $AB : BH (e) :: CD : DF$ ; and  $AG : GH (e) :: CE : EF$ . Likewise  $AG : AH :: (e) CE : CF$ , and  $AH : AB :: CF : CD$ . (f) From whence by equality  $AG : AB :: CE : CD$ . After the same manner  $GH : HB :: EF : FD$ . (g) Therefore the Polygons ABHG, CDFE are similar and alike situate. Which was to be done.

d 2. ax.

e 4. 6.

f 22. 5.

g 1. def. 6.

## P R O P. XIX. Fig. 54.

Like triangles ABC, DEF, are in duplicate ratio of their homologous sides, BC, EF,

(a) Let

(a) Let there be made  $BC : EF :: EF : BG$ , and let  $AG$  be drawn. Because that  $AB : DE$  (b) : :  $AC : EF$  (c) : :  $EF : BG$ , and the angle  $B =$  (d) therefore is the triangle  $ABG = DEF$ . And the triangle  $ABC : ABG ::$  (e)  $BC : BG$ , and (f)  $\frac{BC}{BG} = \frac{ABC}{ABG}$  twice; therefore  $\frac{ABC}{ABG}$  that is,  $\frac{ABC}{DEF}$  (g)  $= \frac{BC}{EF}$  twice. Which was to be demonstrated.

a 11. 6.  
b cor. 4. 6.  
c constr.  
d 15. 6.  
e 1. 6.  
f 10. def. 5.  
g 11. 5.

Coroll.

Hence, If three right-lines ( $BC, EF, BG$ ) are proportional, then as the first is to the third, so is a triangle made upon the first  $BC$ , to a triangle similar and alike described upon the second  $EF$ ; or so is a triangle described upon the second  $EF$ , to a triangle similar and alike described upon the third.

PROP. XX. Plate III. Fig. 56, 57.

Like Polygons  $ABCDE, FGHIK$ , are divided into equal triangles  $ABC, FGH$ , and  $ACD, FHI$ , and  $ADE, FIK$ ; which are equal in number and homologous to the wholes ( $ABC : GH :: ABCDE : FGHIK :: ACD : FHI :: ADE : FIK$ .) And the Polygons  $ABCDE, FGHIK$ , have a duplicate ratio one to the other of what one homologous side  $BC$  hath to the other homologous side  $GH$ .

1. For the angle  $B$  (a)  $= G$ , and  $AB : BC$  (a) : :  $FG : GH$ . (b) Therefore the triangles  $ABC, FGH$ , are equiangular. After the same manner are the triangles  $AED, FIK$  similar. Since therefore the angle  $BCA$  (b)  $= GHF$ , and the angle  $ADE$  (b)  $= FIK$ , and the whole angles  $ACD, GHI$ , and the whole angles  $CDE, HIK$  are (c) equal, there remains the angle  $ACD$  (d)  $= FHI$ , and the angle  $ADC = FIK$ ; (e) from whence also the angle  $AD = HFI$ , therefore the triangles  $ACD, FHI$  are similar. Therefore, &c.

a hyp.  
b 6. 6.

2. Because the triangles  $BCA, GHF$  are like, (f) therefore,  $\frac{BCA}{GHF} = \frac{BC}{GH}$  twice. For the same reason is  $\frac{CAD}{HFI} = \frac{CD}{HI}$  twice; lastly  $\frac{DEA}{IKF} = \frac{DE}{IK}$  twice. Now where-

c hyp.  
d 3. ax.  
e 32. 1.

f 19. 6.

that  $BC : GH$  (g) : :  $CD : HI$  (g) : :  $DE : IK$ ; (h) therefore is the triangle  $BCA : GHF :: CAD : HFI :: DEA : IKF$  (k) : : the polygon  $ABCDE : FGHIK :: BC : GH$  twice.

g hyp. &  
16. 5.  
h cor. 23. 5.  
k 12. 5.

F 4

Coroll.

Coroll.

1. Hence if there are three right-lines proportional, then as the first is to the third, so is a polygon made upon the first to a polygon made on the second similar and alike described; or so is a polygon made upon the second, to a polygon made on the third similar and alike described.

Hence we have a method of enlarging or diminishing any right-lined figure in a ratio given; For if you would make a pentagon quintuple of that pentagon whereof CD is the side, then betwixt AB and 5 AB find out a mean proportional, \* upon this raise a pentagon like to that given, and it shall be quintuple of the pentagon given.

2. Hence also, If the homologous sides of like figures be known, then will the proportion of the figures be evident, viz. by finding out a third proportional.

## P R O P. XXI. Plate III. Fig. 58.

Right-lined figures ABC, DIE, which are similar to the same right-lined figure HFG, are also similar one to the other.

For the angle A (a) = H (a) = D; and the angle C (a) = G (a) = E; and the angle B (a) = F (a) = I. Also (a)  $AB : AC :: HF : HG :: DI : DE$ ; and (a)  $AC : CB :: HG : GF :: DE : EI$ . And  $AB : BC :: HF : FG :: DI : IE$ . Therefore (a) ABC, DIE, are similar. Which was to be demonstrated.

## P R O P. XXII. Fig. 59.

If four right-lines are proportional ( $AB : CD :: EF : GH$ ) the right-lined figures also described upon them being similar and alike situate, shall be proportional ( $ABI : CDK :: EM : GO$ .) And if the right-lined figures described upon the lines, similar and alike situate, be proportional ( $ABI : CDK :: EM : GO$ ) then the right-lines also shall be proportional ( $AB : CD :: EF : GH$ .)

a 19. 6.

1. Hyp.  $\frac{ABI}{CDK} (a) = \frac{AB}{CD}^{twice} = \frac{EF}{GH}^{twice} (a) = \frac{EM}{GO}$   
(b) therefore  $ABI : CDK :: EM : GO$ .

b hyp.

c 20. 6.

d cor. 23. 5

2. Hyp.  $\frac{AB}{CD}^{twice} (a) = \frac{ABI}{CDK} (b) = \frac{EM}{GO} = \frac{EF}{GH} (c)$ . Therefore (d)  $AB : CD :: EF : GH$ , Which was to be demonstrated, Schol.



Schol.

Hence is deduced the manner and reason of multiplying surd quantities, ex. g. Let  $\sqrt{5}$  be to be multiplied into  $\sqrt{3}$ . I say that the product will be  $\sqrt{15}$ . For by the definition of multiplication it ought to be, as  $1 : \sqrt{3} :: \sqrt{5} : \text{the product}$ . Therefore by this q.  $1 : q. \sqrt{3} :: q. \sqrt{5} : q. \text{ of the product}$ . That is,  $1 : 3 :: 5 : \text{the square of the product}$ , therefore the square of the product is 15. Wherefore  $\sqrt{15}$  is the product of  $\sqrt{3}$  into  $\sqrt{5}$ . Which was to be demonstrated.

THEOREM. Plate III. Fig. 50.

If a right-line AB be cut any-wise in E, the rectangle comprehended under the parts AE, EB, is a mean proportional betwixt their squares. Likewise the rectangle comprehended under the whole AB, and one part AE, or EB is a mean proportional betwixt the square of the whole AB and the square of the said part, AE, or EB,

Pet. Herig.

Upon the diameter AB describe a semicircle; from E erect a perpendicular EF, meeting with the periphery in F, join AF, BF.

It's evident that  $AE : EF (a) :: EF : EB$ , (b) therefore  $AEq : EFq :: EFq : EBq$ . (c) that is,  $AEq : AEB :: AEB : EBq$ . Which was to be demonstrated.

a cor. 8. 6.  
b 22. 6.  
c 17. 6.  
d cor. 8. 6.  
e 22. 6.  
f 17. 6.

Moreover  $BA : AF :: (d) AF : AE$ , (e) therefore  $BAq : AFq :: AFq : AEq$ . (f) that is,  $BAq : BAE :: BAE : AEq$ . After the same manner  $ABq : ABE :: ABE : BEq$ . Which was to be demonstrated.

Or thus: suppose  $Z = A + F$ . It is manifest that  $Aq : AF :: (a) A : F :: (a) AF : Fq$ . also  $Zq : ZA :: (a) Z : A :: ZA : Aq$ , and  $Zq : ZF :: (a) Z : F :: ZF : Fq$ .

a 1. 6.

PROP. XXIII. Fig. 34.

Equiangular parallelograms AC, BF, have the ratio one to the other, which is compounded of their sides.  $\left(\frac{AC}{BF} = \frac{AB}{BG} + \frac{BC}{BE}\right)$

Let the sides about the equal angles, B be (a) set in a direct line, and let the Pgr. BH be compleated. Then is

a sch. 19.

the ratio of  $\frac{AC}{BF} (b) = \frac{AC}{BH} + \frac{BH}{BF} (c) = \frac{AB}{BG} + \frac{BC}{BE}$  Which was to be demonstrated.

b 20 def. 5.  
c 1. 6.

Coroll.

Andr.

Tacq. 15. 5

\* 35. 1.

\* 14. 6.

and 1. 6.

Hence, and from 34. 1. it appears, 1. That triangles which have one angle equal (as at C) have a ratio compounded of the ratio's of the right-lines, (A to B, and C to E,) containing the equal angle.

2. That all rectangles, and \* consequently all parallelograms, have their ratio one to the other compounded of the ratio's of base to base, and altitude to altitude. After the like manner you may argue in triangles.

3. From hence is apparent how to give the proportion of triangles and parallelograms.

Let there be two Pgrs. X and Z, whose bases are AC, CB, and altitudes CL, CF. Make  $CL : CF :: CB : O$ , \* then will it be  $X : Z :: AC : O$ .

## P R O P. XXIV. Fig. 55.

In every parallelogram ABCD, the parallelograms EG, HF, which are about the diameter AC, are like to the whole, and also one to the other.

a 29. 1.

b 4. 6.

c 22. 5.

d 1. def. 6.

For the Pgrs. EG, HF, have each of them one angle common with the whole; (a) therefore they are equiangular to the whole, and also one to the other. Also the triangles ABC, AEI, IHC (a) and the triangles ADC, AGI, IFC are equiangular mutually; (b) therefore  $AE : EI :: AB : BC$ , and (b)  $AE : AI :: AB : AC$ , and (b)  $AI : AG :: AC : AD$ , (c) Therefore by equality,  $AE : AG :: AB : AD$ . (d) Therefore the Pgrs. EG, BD are similar, After the same manner are HF, BD similar also. Therefore, &c.

## P R O P. XXV. Fig. 62.

Unto the right-lined figure given ABEDC, to describe another figure P, similar and alike situate, which also shall be equal to another right-lined figure given F.

a 45. 1.

b 44. 1.

c 13. 6.

d 18. 6.

e cor. 20. 6.

f 1. 6.

g 14. 5.

h constr.

(a) Make the rectangle  $AL = ABEDC$ ; (b) also upon BL make the rectangle  $BM = F$ ; betwixt AB and BH (c) find out a mean proportional NO; (e) upon NO (d) make the polygon P similar to the right-lined figure given ABEDC. I say the polygon P so made, shall be equal to F, that was given.

For ABEDC (AL) : P :: AB : BH :: (f) AL : BM. Therefore P (g) = BM (h) = F. Which was to be done.

P R O P.

PROP. XXVI. Plate III. Fig. 60.

If from the parallelogram ABCD, be taken away another parallelogram AGFE, similar to the whole, and alike situate, having also an angle EAG common with it; then is that parallelogram about the same diagonal AC with the whole.

If you deny AC to be the common diagonal, then let AHC be it, cutting EF in H, and let HI be drawn parallel to AE. Then are the Pgrs. EI, DB, (a) similar; (b) therefore  $AE:EH::AD:DC::(c) AE:EF$ , and (d) consequently  $EH=EF$ . (f) Which is absurd.

a 24. 6.  
b 1. def. 6.  
c hyp.  
d 9. 5.  
f 9. ax.

PROP. XXVII. Plate IV. Fig. 1.

Of all parallelograms AD, AG, applied to the same right-line AB, and wanting in figure by the parallelograms CE, KI, similar and alike situate to the Pgr. AD, which is described upon the half line, the greatest is that AD, which is applied to the half line being like to the defect KI.

For because that GE (a) equal GC, if KI which is common be added, (b) then is  $KE=CI$  (c)  $=AM$ ; add CG which is common, (d) then is  $AG=$  to the Gnomon MBL; (e) but the Gnomon  $MBL \supset CE$  (AD) Therefore  $AG \supset AD$ . Which was to be demonstrated.

a 43. 1.  
b 2. ax.  
c 36. 1.  
d 2. ax.  
e 9. ax.

PROP. XXVIII. Fig. 2.

To a right-line given AB, to apply a parallelogram AP, equal to a right-lined figure given C, deficient by a parallelogram ZR, which is similar to another parallelogram given D, \* but it is necessary that the right-lined figure given C, to which the Pgr. to be applied AP must be equal, be not greater than the Pgr. AF, which is applied to half the line, since the defects both of AF, which is apply'd to half the line, and of AP the parallelogram to be applied, must be similar.

\* 27. 6.

Bisect AB in E; upon EB (a) make the Pgr. EG like to the Pgr. D; and (b) let  $EG=C+I$ . (c) Make the Pgr. NT = I, and like to the Pgr. given D, or EG; draw the diameter FB; Make FO = KN, and FQ = KT; thro' O and Q draw the parallels SR, QZ. Then is the Pgr. AP that which was sought.

a 18. 6.  
b sch. 45. 1  
c 25. 6.

For



d const. 5

24. 6.

e const.

f 3. ax.

g 2. ax.

h 43. 1.

For the Pgrs. D, EG, OQ, NT, ZR, are all (d) similar one to the other, and the Pgr.  $EG = (e) NT + C = (e) OQ + C$ ; (f) wherefore  $C =$  to the Gnomon OBQ (g)  $= AO + PG = (b) AO + EP = AP$ . Which was to be done.

## P R O P. XXIX. Plate IV. Fig. 3.

Upon a right-line given A B, to apply a parallelogram AN equal to a right-lined figure given C, exceeding by a Pgr. OP, which shall be like to another Pgr. given D.

a 18. 6.

b 25. 6.

c 3. 1.

Bisect AB in E. Upon EB (a) make a Pgr. EG like to the given one D, and (b) let the Pgr.  $HK = EG + C$ , and like to the given one D, or to EG. Make  $FEL = (c) IH$ ; and (c)  $FGM = IK$ . Thro' L, M, draw the parallels MN and RN; and AR parallel to NM. Produce ABP, GBO; draw the diameter FBN. Then is AN the parallelograms required.

d const.

e 24. 6.

f const.

g 3. ax.

h 36. 1.

k 43. 1.

l 2. and 1.

ax.

For the Pgrs. D, HK, LM, EG, are (d) similar; (e) therefore the Pgr. OP is similar to the Pgr. LM, or D. Also  $LM (f) = HK (f) = EG + C$ . (g) Therefore  $C =$  to the Gnomon ENG. But  $AL (b) = LB (k) = BM$ ; (l) therefore  $C = AN$ . Which was to be done.

## P R O P. XXX. Fig. 5.

To cut a finite right-line given AB, according to extreme and mean ratio ( $AB : AG :: AG : GB$ .)

a 11. 2.

b 17. 6.

(a) Cut AB in G, in such wise that  $AB \times BG = AG^2$ . (b) Then  $BA : AG :: AG : GB$ . Which was to be done.

## P R O P. XXXI. Fig. 6.

In right-angled triangles BAC, any figure BF described upon the side BC, subtending the right-angle BAC, is equal to the figures BG, AL, which are similar and alike situated to the former BF, and described upon the sides BA, AC containing the right-angle.

a cor. 8. 6.

b cor. 20. 6.

c 24. 5.

d sch. 14. 5.

e 22. 6.

f 24. 5.

g sch. 14. 5.

h 47. 1.

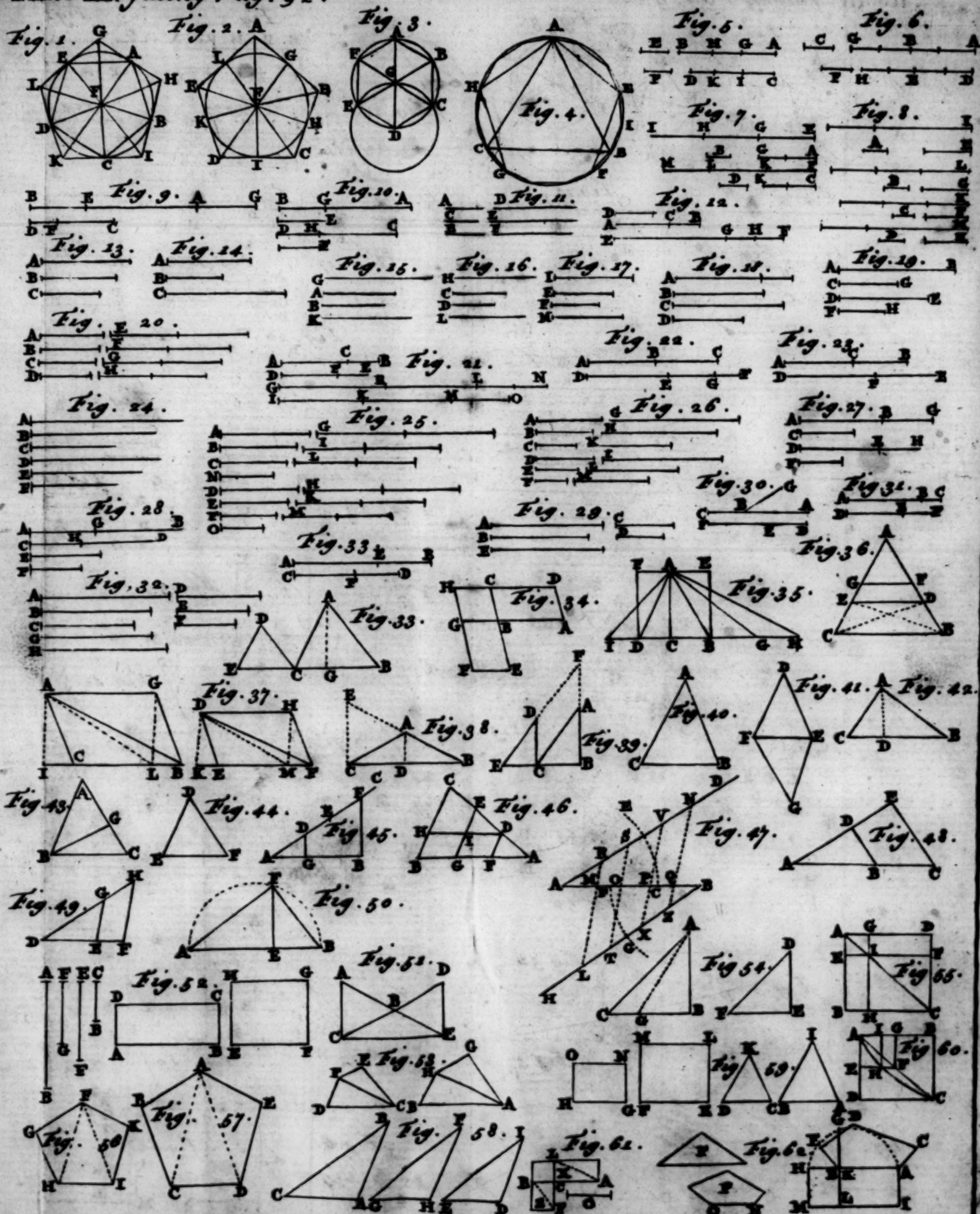
From the right-angle BAC let fall the perpendicular AD. Because  $DC : CA :: (a) CA : CB$ , therefore  $AL : BF :: DC : CB$ . Also, because  $DB : BA :: (a) BA : BC$ , (b) therefore  $BG : BF :: DB : BC$ ; (c) therefore  $AL + BG : BF :: DC + DB (BC) : BC$ . (d) Therefore  $AL + BG = BF$ . Which was to be demonstrated.

Or thus:  $BG : BF :: (e) BAq : BCq$ : And (e)  $AL : BF :: ACq : BCq$ , (f) therefore  $BG + AL : BF :: BAq + ACq : BCq$ . (g) Therefore whereas  $BAq + ACq = (b) BCq$ ; (b) thence is  $BG + AL = BF$ . Which was to be demonstrated.

Corol.











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Coroll.

From this proposition you may learn how to add or subtract any like figures, by the same method that is used in adding and subtracting of squares, in Schol. 47. 1.

PROP. XXXII. Plate IV. Fig. 4.

If two triangles ABC, DCE having two sides proportional to two ( $AB : AC :: DC : DE$ ) be so compounded or set together at one angle ACD, that their homologous sides are also parallel ( $AB$  to  $DC$ , and  $AC$  to  $DE$ ), then the remaining sides of those triangles ( $BC$ ,  $CE$ ) shall be found placed in one strait-line.

For the angle  $A (a) = ACD (a) = D$ , and  $AB : AC (b) :: DC : DE$ , ( $c$ ) therefore the angle  $B = DCE$ . Therefore the angle  $B + A (d) = ACE$ ; but the angle  $B + A + ACB (e) = 2$  right, ( $f$ ) therefore the angle  $ACE + ACB = 2$  right; ( $g$ ) therefore  $BCE$  is a right-line. Which was to be demonstrated.

a 29. 1.  
b hyp.  
c 6. 6.  
d 2. ax.  
e 32. 1.  
f 1. ax.  
g 14. 1.

PROP. XXXIII. Fig. 7, 8.

In equal circles  $DBCA$ ,  $HFGP$ , the angles  $BDC$ ,  $FHG$ , have the same ratio with the peripheries  $BC$ ,  $FG$ , on which they stand; whether the angles be set at the centers (as  $BDC$ ,  $FHG$ ) or at the circumferences,  $A$ ,  $E$ : And so likewise have the Sectors  $BDC$ ,  $FHG$ .

Draw the right-lines  $BC$ ,  $FG$ . Make  $CI = CB$ , and  $GL = FG = LP$ , and join  $DI$ ,  $HL$ ,  $HP$ .

The arch  $BC (a) = CI$ , ( $a$ ) also the arches  $FG$ ,  $GL$ ,  $LP$ , are equal; ( $b$ ) therefore the angle  $BDC = CDI$ , ( $b$ ) and the angle  $FHG = GHL = LHP$ . Therefore the arch  $BI$  is the same multiple of the arch  $EC$ , as the angle  $BDI$  is of the angle  $BDC$ . And in like manner is the arch  $FP$ , the same multiple of the arch  $FG$ , as the angle  $FHP$  is of the angle  $FHG$ . But if the arch  $BI \sqsubset, =, \sqsupset FP$ , ( $c$ ) then likewise is the angle  $BDI \sqsubset, =, \sqsupset FHP$ . Therefore as the arch  $BC : FG :: (d)$  the angle  $BDC : FHG$ .

a 28. 3.  
b 27. 3.  
c 27. 3.  
d 6. def, 5.  
e 15. 5.  
f 20. 3.  
g 27. 3.  
h 24. 3.  
k 4. 1.  
l 2. ax.

$$(e) :: \frac{BDC}{2} : \frac{FGH}{2} (f) :: A : E.$$
 Which was to be dem.

Moreover, the angle  $BMC (g) = CNI$ ; ( $b$ ) and therefore the segment  $BCM = CIN$ . ( $k$ ) Also the triangle  $BDC = CDI$ ; ( $l$ ) wherefore the sector  $BDCM = CDIN$ . After

After the same manner are the sectors FHG, GHL, LHP equal one to the other. Therefore since accordingly as the arch BI  $\overset{\frown}{\text{BI}}$ ,  $=$ ,  $\overset{\frown}{\text{FGP}}$ , so is likewise the sector BDI  $\overset{\frown}{\text{BDI}}$ ,  $=$ ,  $\overset{\frown}{\text{FHP}}$ ; (m) thence it will be as the sector BDC: FHG: : the arch BC: FG, Which was to be demonstrated.

m 6. def. 5.

Coroll.

a 11. 5.

1. Hence, As sector (a) is to sector, so is angle to angle.
2. The angle BDC in the center, is to four right-angles, as the arch BC, on which it stands, to the whole circumference.

For as the angle BDC is to a right-angle, so is the arch BC to a quadrant. Therefore BDC is to four right-angles, as the arch BC is to four quadrants, that is, to the whole circumference. Also, as the angle A: 2 right: : the arch BC: periphery.

3. Hence, the arches IL, BC, (Fig. 9.) of unequal circles which subtend equal angles, whether at the centers, as IAL and BAC, or at the periphery, are like.

For IL: periph.: : angle IAL (BAC): 4 right. Also, Arch BC: periph.: : angle BAC: 4 right. Therefore IL: periph.: : BC: periph. And consequently the arches IL and BC are similar. Whence

4. Two semidiameters AB, AC, cut off like arches IL, BC from concentric peripheries.

**The End of the sixth Book.**

THE



# The SEVENTH BOOK OF EUCLID'S ELEMENTS.

## Definitions.

I. **U**Nity is that, by which every thing that is, is called One.

II. Number is a multitude composed of units.

III. Part is a number of a number; the lesser of the greater, when the lesser measureth the greater.

*Every part is denominated from the number, by which it measures the number whereof it is a part; as 4 is called the third part of 12, because it measures 12 by 3.*

IV. But when the lesser number does not measure the greater, then the lesser is call'd, not a part, but parts of the greater.

*All parts whatsoever are denominated from these two numbers, by which the greatest common measure of the two numbers measures each of them; as 10 is said to be two thirds of the number 15; because the greatest common measure, which is 5, measures 10 by 2, and 15 by 3.*

V. A multiple is a greater number compared with a lesser, when the lesser measures the greater.

VI. An even number is that which may be divided into two equal parts.

VII. But an odd number is that which cannot be divided into two equal parts; or that which differeth from an even number by unity.

VIII. A number evenly even, is that which an even number measureth by an even number.

IX. But a number evenly odd, is that which an even number measureth by an odd number.

X. A number only oddly odd, is that which an odd number measureth by an odd number.

XI.

XI. A prime (or first) number is that which is measured only by unity.

XII. Numbers prime the one to the other, are such as only unity doth measure, being their common measure.

XIII. A composed number is that which some certain number measureth.

XIV. Numbers composed the one to the other, are those, which some number, being a common measure to them both, doth measure.

*In this, and the preceding definition, unity is not a number.*

XV. One number is said to multiply another when the number multiplied is so often added to it self, as there are units in the number multiplying, and another number is produced,

*Hence in every multiplication unity is to the multiplier, as the multiplicand is to the product.*

Obs. That many times, when any numbers are to be multiplied (as A into B) the conjunction of the letters denotes the product: So  $AB = A \times B$ , and  $CDE = C \times D \times E$ .

XVI. When two numbers multiplying themselves produce another, the number produced is called a plane number; and the numbers which multiplied one another, are called the sides of it: So  $2(C) \times 3(D) = 6 = CD$  is a plane number.

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a solid number; and the numbers multiplying one another, are called the sides thereof: So  $2(C) \times 3(D) \times 5(E) = 30 = CDE$  is a solid number.

XVIII. A square number is that which is equally equal; or, which is contained under two equal numbers. Let A be the side of a square; the square is thus noted, AA, or Aq.

XIX. A Cube is that number which is equally equal equally; or which is contained under three equal numbers. Let A be the side of a Cube; the Cube is thus noted, AAA, or Ac.

*In this definition, and the three foregoing, unity is number.*

XX. Numbers are proportional, when the first is the same multiple of the second, as the third is of the fourth; or, the same part; or, when a part of the first number measures the second, and the same part of the third

third measures the fourth, equally: and *vice versa*. So  
 $A : B :: C : D$ . that is,  $3 : 9 :: 5 : 15$ .

XXI. Like plane, and solid numbers, are those which have their sides proportional: *Namely, not all the sides, but some.*

XXII. A perfect Number is that which is equal to its own parts.

As 6, and 28. But a number that is less than it's parts is called an Abounding number, and one which is greater, a Diminutive: so 12 is an abounding, 15 a diminutive number.

XXIII. One number is said to measure another, by a third number, which when it either multiplies, or is multiplied by the measuring number, produces the number measured.

In Division, unity is to the quotient, as the divisor is to the dividend. Note, that a number placed under another

with a line between them, signifies division: So  $\frac{A}{B} = A$  divided by  $B$ , and  $\frac{CA}{B} = CA$  divided by  $B$ .

Those two numbers are called the Terms or Roots of a Proportion, than which lesser cannot be found in the same proportion.

*Postulates, or Petitions.*

1. **T**hat numbers equal or multiple to any number may be taken at pleasure.
2. That a number greater than any other whatsoever may be taken.
3. That Addition, Subtraction, Multiplication, Division, and the Extractions of Roots or sides of square and cube numbers, be also granted as possible.

*Axioms.*

1. **W**hatsoever agrees with one of many equal numbers, agrees likewise with the rest.
2. Those parts that are the same to the same part, or parts, are the same among themselves.
3. Numbers that are the same parts of equal numbers, or of the same number, are equal among themselves.

G

4. Those



4. Those numbers of which the same number or equal numbers, are the same parts, are equal amongst themselves.

5. Unity measures every number by the units that are in it, that is, by the same number.

6. Every number measures it self by unity.

7. If one number multiplying another, produces a third, the multiplier shall measure the product by the multiplied; and the multiplied shall measure the same by the multiplier.

Hence, No prime number is either a plane, solid, square, or cube number.

8. If one number measures another, that number by which it measureth shall measure the same by the units that are in the number measuring, that is, by the number it self that measures.

9. If a number measuring another, multiply that by which it measureth, or be multiplied by it, it produceth the number which it measureth.

10. How many numbers soever any number measureth, it likewise measureth the numbers composed of them.

11. If a number measures any number, it also measureth every number which the said number measureth.

12. A number that measures the whole and a part taken away, doth also measure the residue.

# PROP. I.

A.....E...G. B 8 5 3  
C...F...D.  $\frac{2}{1}$   $\frac{5}{3}$   $\frac{1}{2}$   
H---

Two unequal numbers AB, CD, being given, if the lesser CD, be continually taken from the greater AB (and the residue

EB from CD, &c.) by an alternate subtraction, and the number remaining never measures the precedent, till unity GB be taken; then are the numbers which were given AB, CD, prime the one to the other.

- If you deny it, let AB, CD, have a common measure, namely the number H; therefore H measuring CD, doth  
 a 11. ax. 7. (a) also measure AE; and (b) consequently the remainder EB; (a) therefore it likewise measures CF, and (b) so the  
 b 12. ax. 7. remainder FD; (a) therefore it also measures EG. But it measured the whole EB, and (b) therefore it must measure that which remaineth GB, that is, a number measures unity. (c) Which is absurd.  
 c 9. ax. 1.

PROP

PROP. II.

Two numbers AB, 9 6  
 CD being given, not A ..... E ..... B 15 9 6  
 prime the one to the other, 6 3  
 to find out their greatest C ..... F ... D 8 3 1  
 common measure FD. G ---

Take the lesser number CD from the greater AB as often as you can. If nothing remains, (a) it is manifest that CD is the greatest common measure. But if there remains something (as EB) then take it out of CD, and the residue FD out of EB, and so forward till some number (FD) measure the said EB, (b) for this will be, before you come to unity; FD shall be the greatest common measure.

For FD (c) measures EB; and (d) therefore also CF; and (e) consequently the whole CD; (d) therefore likewise AE; and so measures the whole AB. Wherefore it is evident that FD is a common measure. If you deny it to be the greatest, let there be a greater (G); then whereas G measureth CD, it (d) must likewise measure AE, (e) and the residue EB, (d) as also CF, (e) and by consequence the residue FD, (g) the greater the less. (b) Which is absurd.

Coroll.

Hence, a number that measures two numbers, does also measure their greatest common measure.

PROP. III.

Three numbers being given, A, B, A ..... 12.  
 C, not prime one to the other, to find out B ..... 8  
 their greatest common measure E. D .... 4.

Find out D, the greatest common C ..... 6  
 measure of the two numbers A, B. E .. 2

If D measures C the third, it is F ---  
 clear that D is the greatest common  
 measure of all the three numbers. If D does not mea-  
 sure C, at least D and C will be composed the one to  
 the other, by the Coroll. of the Proposition preceding.  
 Therefore let E be the greatest common measure of the  
 said numbers D and C, and it shall be the number which  
 was required.

a 6. ax. 7.

b 1. 7.

c constr.  
 d 11. ax. 7.  
 e 12. ax. 7.

d 11. ax. 7.  
 e 12. ax. 7.  
 g suppos.  
 h 9. ax. 1.

a const.

b 11. ax. 7.

c cor. 1 7.

d suppos.

e 9. ax. 1.

For E (a) measures C and D, and D measures A and B; therefore (b) E measures each of the numbers A, B, C: neither shall any greater (F) measure them; for if you affirm that, (c) then F measuring A and B, does likewise measure D their greatest common measure; and in like manner, F measuring D and C, does also measure E (c) their greatest common measure, (d) the greater the less. (e) Which is absurd.

Coroll.

Hence, a number that measures three numbers, does also measure their greatest common measure.

## PROP. IV.

A.....6

B.....7

B.....18

B.....9

a 4. def. 7.

b 3. def. 7.

c 4. def. 7.

Every less number A is of every greater B either a part or parts. If A and B be prime to one another, (a) A shall be as many parts of the number B, as there are units in A (as  $6 = \frac{2}{3}$  of 7). But if A measures B, it is (b) plain that A is a part of B (as  $6 = \frac{1}{3}$  of 18.) Lastly, if A and B be otherwise composed to one another, (c) the greatest common measure shall determine how many parts A does contain of B; as  $6 = \frac{2}{3}$  of 9.

## PROP. V.

A.....6

D.....4

6

6

4

4

B.....G.....C 12 E.....H.....F 8.

If a number A be a part of a number BC, and another number D the same part of another number EF; then both the numbers together (A+D) shall be the same part of both the numbers together (BC+EF,) which one number A is of one number BC.

For if BC be resolved into its parts BG, GC, equal to A; and EF also into its parts EH, HF, equal to D; (a) the number of parts in BC shall be equal to the number of parts in EF. Therefore since A+D (b) = BG+EH = GC+HF, thence A+D shall be as often in BC+EF, as A is in BC. Which was to be demonstrated.

a hyp.

b const. 6

c ax. 1.

Or



Or thus. Let  $a = \frac{x}{2}$ , and  $b = \frac{y}{2}$ , then (c)  $2a = x$ , and  $2b = y$ , therefore  $2a + 2b = x + y$ , therefore  $a + b = \frac{x+y}{2}$  Which was to be demonstrated.

PROP. VI.

If a number AB be divided into its parts AG, GB; and another number DE into its parts DH, HE. The number of parts in both AB, DE, is equal by supposition; since then AG (a) is the same part of the number C, that DH is of the number F, AG + DH (b) shall be the same part of the compounded number C + F, that one number AG is of one number C. (b) In like manner GB + HE is the same part of the said C + F, that one number GB is of one number C. (c) Therefore AB + DE is the same parts of C + F, that AB is of C. Which was to be demonstrated.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The number of parts in both AB, DE, is equal by supposition; since then AG (a) is the same part of the number C, that DH is of the number F, AG + DH (b) shall be the same part of the compounded number C + F, that one number AG is of one number C. (b) In like manner GB + HE is the same part of the said C + F, that one number GB is of one number C. (c) Therefore AB + DE is the same parts of C + F, that AB is of C. Which was to be demonstrated.

Or thus. Let  $a = \frac{2}{3}x$ , and  $b = \frac{2}{3}y$ , and  $x + y = g$ , then, because  $3a = 2x$ , and  $3b = 2y$ , is  $3a + 3b = 2x + 2y = 2g$ , therefore  $a + b = \frac{2}{3}g = \frac{2}{3}(x + y)$ .

PROP. VII.

If a number AB be divided into its parts AG, GB; and another number DE into its parts DH, HE. The number of parts in both AB, DE, is equal by supposition; since then AG (a) is the same part of the number C, that DH is of the number F, AG + DH (b) shall be the same part of the compounded number C + F, that one number AG is of one number C. (b) In like manner GB + HE is the same part of the said C + F, that one number GB is of one number C. (c) Therefore AB + DE is the same parts of C + F, that AB is of C. Which was to be demonstrated.

(a) Let EB be the same part of the number GC that AB is of CD, or AE of CF, (b) therefore AE + EB is the same part of CF + GC that AE is of CF, or AB of CD; (c) therefore GF = CD. Take away CF common to both, and (d) there remains GC = FD. (e) Wherefore

G 3

fore

c 2. ax. 1.

a hyp.

b 5. 7.

c 2. ax. 7.

a 1 post. 7.

b 5. 7.

c 6. ax. 1.

d 3. ax. 1.

e 2. ax. 7.

fore EB is the same part of the residue FD (GC) that the whole AB is of the whole CD. *Which was to be dem.*

Or thus. Let  $a + b = x$ ; and  $c + d = y$ ; and  $x = 3y$ , in like manner as  $a = 3c$ ; I say  $b = 3d$ . (f) For  $3c + 3d = 3y = x$  (g)  $= a + b$ , take away from both  $3c$  (g)  $= a$ , and there remains  $3d = b$ . *Which was to be dem.*

f 1. 2.

g hyp.

## P R O P. VIII.

$$\begin{array}{ccccccc} 6 & 2 & 4 & 2 & 2 & & \\ A \dots H \dots G \dots E \dots L \dots B & 16 & & & & & \\ & 18 & & 6 & & & \\ C \dots \dots \dots F \dots \dots D & 24 & & & & & \end{array}$$

If in a number AB there be the same parts of a number CD, that a part taken away AE, is of a part taken away CF; the residue also EB shall be the same parts of the residue FD, that the whole AB is of the whole CD.

a 3. ax.

b constr.

c 3. ax. 1.

d 7. 7.

Divide AB into AG, GB, parts of the number CD; also AE into AH, HE, parts of the number CF; and take  $GL = AH = HE$ , (a) wherefore  $HG = EL$ . And because (b)  $AG = GB$ , (c) therefore  $HG = LB$ . Now whereas the whole AG is the same part of the whole CD that the part taken away AH is of the part taken away CF, (d) the residue HG or EL shall be the same part also of the residue FD that AG is of CD. In like manner, because GB is the same part of the whole CD, that HE or GL are of CF, (d) therefore the residue LB shall be the same part of the residue FD that GB is of the whole CD. Therefore  $EL + LB$  (EB) is the same parts of the residue FD, that the whole AB is of the whole CD. *Which was to be demonstrated.*

e 9. ax. 7.

f 1. 2.

g 1. ax. 1.

h hyp.

k 3. ax. 1.

l 8. ax. 7.

Or thus more easily. Let  $a + b = x$ , and  $c + d = y$ . Also  $y = \frac{2}{3}x$  as well as  $c = \frac{2}{3}a$ ; or, (e) which is the same  $3y = 2x$ ; and  $3c = 2a$ . I say  $d = \frac{2}{3}b$ . For  $3c + 3d$  (f)  $= 3y = 2x$  (f)  $= 2a + 2b$ . (g) Therefore  $3c + 3d = 2a + 2b$ ; take away from each  $3c$  (h)  $= 2a$ , and (k) there remains  $3d = 2b$ ; (l) therefore  $d = \frac{2}{3}b$ . *Which was to be demonstrated.*

## P R O P. IX.

$$\begin{array}{ccccccc} A \dots 4 & & & & & & \\ B \dots 4 \dots G \dots C & 8 & & & & & \\ & D \dots 5 & & & & & \\ & 5 & & 5 & & & \\ E \dots H \dots F & 10 & & & & & \end{array}$$

If a number A be a part of a number BC, and another number D the same part of another number EF; then alternately what part or parts the first A is of the third D, the same part or parts shall the second BC be of the fourth EF.

A is supposed  $\supset$  D, therefore let BG, GC, and EH, HF, parts of the numbers BC, EF be equal; BG and GC to A; and EH, HF to D. The multitude of parts is put equal in both. But it is clear that BG is (a) the same part or parts of EH, that GC is of HF; (b) wherefore BC (BG + GC) is the same part or parts of EF (EH + HF) that BG alone (A) is of EH alone (D.) Which was to be demonstrated.

a 1. ax. 7.  
 & 4. 7.  
 b 5 or 6. 7.

Or thus. Let  $a = \frac{b}{3}$ , and  $c = \frac{d}{3}$ ; or  $3a = b$ , and  $3c = d$ , then  $\frac{c}{a} = \left( \frac{3c}{3a} = \right) \frac{d}{b}$ .

\* 15. 5.

P R O P. X.

If a number AB be parts of A...G...B 4  
 a number C, and another number DE, the same parts of another number F, then alternately, what parts or part the first AB is of the third DE, the same parts or part shall the second C be of the fourth F.

AB is taken  $\supset$  DE, and C  $\supset$  F. Let AG, GB, and DH, HE, be parts of the numbers C and F, viz. as many in AB as in DE. It is manifest that AG is the same part of C, that DH is of F; (a) whence alternately AG is of DH, and likewise GB of HE, and (b) so conjointly AB of DE the same part, or parts, that C is of F. Which was to be demonstrated.

a 9. 7.  
 b 5 & 9. 7.

Or thus. Let  $a = \frac{2b}{3}$ , and  $c = \frac{2d}{3}$ ; or  $3a = 2b$ , and  $3c = 2d$ . Then is  $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$ .

P R O P. XI.

If a part taken away AE be to a part taken away CF, as the whole AB is to the whole CD, the residue also EB shall be to the residue FD, as the whole AB is to the whole CD.

G 4

First,



a 4. 7.  
b 20. def. 7.  
c 7. or 8. 7.

First, let AB be  $\supset$  CD; (a) then AB is either a part or parts of the number CD; and likewise AE is (b) the same part or parts of CF; (c) therefore the residue EB is the same part or parts of the residue FD that the whole AB is of the whole CD, (b) and so  $AB : CD :: EB : FB$ . But if AB be  $\sqsubset$  CD, then according to what is already shewn, will  $CD : AB :: FD : EB$ , therefore by inversion  $AB : CD :: EB : FD$ . *W.W.D.*

## PROP. XII.

A, 4. C, 2. E, 3.  
B, 8. D, 4. F, 6.

*If there be numbers, how many soever, proportional (A : B :: C : D :: E : F;) then as one of the antecedents A is to one of the consequents B, so shall all the antecedents (A + C + E) be to all the consequents (B + D + F.)*

First, let A, C, E, be  $\supset$  B, D, F; then, because of the same proportions, (a) shall A be the same part or parts of B that C is of D; (b) and likewise conjointly A + C shall be the same part or parts of B + D, that A alone is of B alone. In the like manner A + C + E is the same part or parts of B + D + F that A is of B. (c) Therefore  $A + C + E : B + D + F :: A : B$ . But if A, C, E, be put greater than B, D, F, the same thing may be shewn by inversion. *W.W.D.*

## PROP. XIII.

A, 3. C, 4.  
B, 9. D, 12.

*If there be four numbers proportional (A : B :: C : D) then alternately they shall also be proportional, (A : C :: B : D.)*

First, let A and C be  $\supset$  B and D, and A  $\supset$  C. By reason of the same proportion (a) A shall be the same part or parts of B, that C is of D. (b) Therefore alternately A is the same part or parts of C, that B is of D; and so  $A : C :: B : D$ . But if A be  $\sqsubset$  C, and A and C supposed  $\sqsubset$  B and D, it will come to the same thing by inverting the proportions. *W.W.D.*

## PROP. XIV.

A, 9. D, 6.  
B, 6. E, 4.  
C, 3. F, 2.

*If there be numbers, how many soever, A, B, C, and as many more equal to them in number, which may be compared two and two in the same proportion (A : B :: C : D)*

$D : E$  and  $B : C :: E : F$ ;) they shall also by equality, be in the same proportion ( $A : C :: D : F$ )

For because  $A : B :: D : E$ , (a) therefore alternately is  $A : D :: B : E :: (a) C : F$ ; (a) therefore again, by permutation,  $A : C :: D : F$ . Which was to be demonstrated,

a 13. 7.

PROP. XV.

If an unite measure any number B, and another number D equally measure some other number E; alternately also shall an unite measure the third number D, as often as the second B doth the fourth E.

For seeing 1 is the same part of B, that D is of E; (a) therefore alternately shall 1 be the same part of D, that B is of E. Which was to be demonstrated.

1. D..2  
B...3. E.....6.

a 9. 7.

PROP. XVI.

If two numbers A, B, mutually multiplying themselves, produce any numbers AB, BA; the numbers produced AB, and BA, shall be equal to one to the other.

B, 4. A, 3.  
A, 3. B, 4.  
AB, 12. BA, 12.

For because  $AB = A \times B$ , (a) therefore shall 1 be as often in A, as B in AB, (b) and by consequence alternately 1 shall be as often in B as A in AB. But because  $BA = B \times A$ , (a) therefore shall 1 be as often in B, as A in BA, therefore as often as 1 is in AB, so often is 1 in BA, and (c) so  $AB = BA$ . Which was to be demonstrated.

a 15. def. 7.  
b 15. 7.

c 4. ax. 7.

PROP. XVII.

If a number A multiplying two numbers B, C, produce other numbers AB, AC; the numbers produced of them shall be in the same proportion that the numbers multiplied are. ( $AB : AC :: B : C$ .)

A, 3.  
B, 2. C, 4.  
AB, 6. AC, 12.

For since  $AB = A \times B$  (a) therefore shall 1 be as often in A as B in AB. (a) Likewise because  $AC = A \times C$  shall 1 be as often in A, as C in AC, and so also B as often in AB as C in AC; (b) wherefore  $B : AB :: C : AC$ , (c) and therefore also alternately  $B : C :: AB : AC$ . Which was to be demonstrated,

a 15. def. 7

b 20. def. 7.  
c 13. 7.

ROP.

## P R O P. XVIII.

$$\begin{array}{r} \text{C, 5.} \\ \text{A, 3.} \\ \hline \text{AC, 15.} \end{array} \quad \begin{array}{r} \text{C, 5.} \\ \text{B, 9.} \\ \hline \text{BC, 45.} \end{array}$$

If two numbers AB, multiplying any number C, produce other numbers AC, BC; the numbers produced of them shall be in the same proportion that the numbers multiplied are. ( $A : B :: AC : BC$ .)

a 16. 7.  
b 17. 7.

For (a)  $AC = CA$ , and  $BC (a) = CB$ ; so the same C multiplying A and B produceth AC and BC; (b) therefore  $A : B :: AC : BC$ . Which was to be demonstrated.

Schol.

Hence is deduced the vulgar manner of reducing fractions ( $\frac{2}{3}, \frac{7}{9}$ ) to the same denomination. For multiply 9 both by 3 and 5, and they produce  $\frac{27}{27} = \frac{1}{1}$ ; because by this  $3 : 5 :: 27 : 45$ . Likewise multiply 5 by 7 and 9, there arises  $\frac{35}{45} = \frac{7}{9}$ ; because  $7 : 9 :: 35 : 45$ .

## P R O P. XIX.

$$\begin{array}{r} \text{A, 4. B, 6. C, 8. D, 12.} \\ \text{AD, 48. BC, 48.} \end{array}$$

If there be four numbers in proportion ( $A : B :: C : D$ .) the rectangle or number produced of the first and fourth (AD) is equal to the rectangle or number which is produced of the second and third (BC.) And if the number which is produced of the first and fourth (AD) be equal to that produced of the second and third (BC) those four numbers shall be in proportion ( $A : B :: C : D$ .)

a 17. 7.  
b hyp.  
c 18. 7.  
d 9. 5.  
e hyp.  
f 17. 5.  
g 17. 7.  
h 18. 7.  
k 11. 5.

1. Hyp. For  $AC : AD (a) :: C : D (b) :: A : B (c) :: AC : BC$ ; (d) therefore  $AD = BC$ . Which was to be demonstrated.

2. Hyp. Because (e)  $AD = BC$ , therefore  $AC : AD (f) :: AC : BC$ . But  $AC : AD (g) :: C : D$ , and  $AC : BC (h) :: A : B$ ; (i) therefore  $C : D :: A : B$ . Which was to be demonstrated.



PROP. XX.

If there are three numbers in proportion ( $A : B :: B : C$ ) the number contained under the extremes ( $AC$ ) equal to the square made of the middle ( $BB$ ). And if the number contained under the extremes be equal to that ( $Bq$ ) produced of the middle, those three numbers shall be in proportion  $A : B :: B : C$ .

| A.      | B.      | C. |
|---------|---------|----|
| 4.      | 6.      | 9. |
| AC, 36. | BB, 36. |    |
|         | D, 6.   |    |

1. Hyp. For take  $D=B$ , (a) therefore  $A : B :: D (B)$  (b) wherefore  $AC=BD$ , (a) or  $BB$ . Which was to be demonstrated.

a 1. ax. 7.  
b 19. 7.

2. Hyp. Because  $AC (c) = BD$ , (b) therefore  $A : B :: (B) : C$ . Which was to be demonstrated.

c hyp.  
d 19. 7.

PROP. XXI.

Numbers  $AB, CD$ ,  $A...G..B 5$ .  $E.....10$ .  
the least of all that  $C..H..D 3$ .  $F.....6$ .  
be the same proportion

with them ( $E, F$ ) equally measure the numbers  $E, F$ , having the same proportion with them; the greater  $AB$  the greater  $E$ , and the lesser  $CD$ , the lesser  $F$ .

For  $AB : CD (a) :: E : F$ ; (b) therefore alternately  $AB : E : CD : F$ ; (c) therefore  $AB$  is the same part or parts of  $E$  that  $CD$  is of  $F$ ; but parts it cannot be, for if so, let  $AG, GB$ , be parts of the number  $E$ ; and  $CH, HD$ , parts of the number  $F$ , (c) therefore  $AG : E :: CH : F$ , and by inversion  $AG : CH (d) :: E : F$ , (e) therefore  $AB : CD$ ; therefore  $AB, CD$ , are not the least in the proportion; which is contrary to the hypothesis. Therefore, &c.

a hyp.  
b 13. 7.  
c 20. aef.

d 13. 7.  
e hyp.

PROP. XXII.

If there are three numbers  $A, B, C$ ;  $A, 4$ .  $D, 12$ .  
other numbers equal to them in number  $B, 3$ .  $E, 8$ .  
 $D, E, F$ ; which may be compared  $C, 2$ .  $F, 6$ .  
two and two in the same proportion; and if also the proportion of them be perturbed ( $A : B :: E : F$ , and  $B : C :: D : E$ .) then by equality they shall be in the same proportion ( $A : C :: D : F$ .)

For

a hyp.

b 19. 7.

c 1 ax. 1.

d 19. 7.

For because  $A : B (a) :: E : F$ , therefore shall  $AF = BE$ ; and because  $B : C :: (a) D : E$ , (b) therefore  $BE = CD$ , (c) and consequently  $AF = CD$ . (d) Therefore  $A : C :: D : F$ . Which was to be demonstrated.

## P R O P. XXIII.

A, 9. B, 4.

C --- D ---

E - - -

Numbers prime the one to the other, A, B, are the least of all numbers that have the same proportion with them.

a 21. 7.

b 23. def. 7.

c 15. 7.

If it be possible, let C and D be less than A and B, and in the same proportion; (a) therefore C measures A equally as D measures B, suppose by the same number E; and so C shall be (b) as often in A as 1 is in E, (c) likewise alternately, E as often in A as 1 in C. By the like reasoning as many times as 1 is in D, so many times shall E be in B. Therefore E measures both A and B; which consequently are not prime the one to the other, contrary to the hypothesis.

## P R O P. XXIV.

A, 9. B, 4.

C ---

D --- E - -

Numbers A, B, being the least of all that have the same proportion with them, are prime the one to the other.

a 9. ax. 7.

b 17. 7.

If it be possible, let A and B have a common measure C; and let the same measure A by D, and B by E; (a) therefore  $CD = A$ , (b) and  $CE = B$ . (b) Wherefore  $A : B :: D : E$ . But D and E are lesser than A and B, as being but parts of them. Therefore A and B are not the least in their proportion, against the Hypothesis.

## P R O P. XXV.

A, 9. B, 4.

C, 3. D - -

number B.

If two numbers A, B, are prime the one to the other, the number C measuring one of them A, shall be prime to the

a 11. ax. 7.

number B. For if you affirm any other D to measure the numbers B and C, (a) then D measuring C does also measure A, and consequently A and B are not prime the one to the other. Which is against the Hypothesis.

P R O P.

PROP. XXVI.

If two numbers A, B, are prime to any number C, the number also produced of them AB, shall be prime to the same C.

|                |        |
|----------------|--------|
| A, 5.          | C, 8.  |
| B, 3.          |        |
| <u>AB, 15.</u> | E ---- |
|                | F ---- |

If it be possible, let the number E be a common measure to AB, and C; and let  $\frac{AB}{E}$  be = F; (a) thence AB = EF; (b) wherefore also F : A :: B : F, But because A is prime to C, which E measures, (c) therefore E and A are prime to one another, (d) and so least in their own proportion, (e) and consequently they must measure B and F; namely F shall measure B, and A shall measure F. Therefore seeing E measures both B and C, they shall not be prime to one another: *Contrary to the Hypothesis.*

a 9. ax. 7.

b 19. 7.

c 25. 7.

d 23. 7.

e 21. 7.

PROP. XXVII.

If two numbers A, B, are prime to one another, that also which is produced of one of them (Aq) shall be prime to the other B.

|         |       |
|---------|-------|
| A, 4.   | B, 5. |
| Aq, 16. |       |
| D, 4.   |       |

Take D = A; therefore both D, and A are prime to B; (b) therefore AD or Aq is prime to B. Which was to be demonstrated.

a 1. ax. 7.

b 26. 7.

PROP. XXVIII.

If two numbers A, B, are prime to two numbers C, D, each to either of both, the numbers also produced by multiplying them AB, CD, shall be prime to one another.

|                |               |
|----------------|---------------|
| A, 5.          | C, 4.         |
| B, 3.          | D, 2.         |
| <u>AB, 15.</u> | <u>CD, 8.</u> |

For because A and B are prime to C, (a) therefore shall AB also be prime to the same. And for the same reason shall AB be prime to D. (b) Therefore AB is prime to CD. Which was to be demonstrated.

a 26. 7.

b 26. 7.

PROP. XXIX.

If two numbers A, B, are prime to one another, and each multiplied by itself produces another number (Aq, and Bq;) then the numbers pro-

|         |        |
|---------|--------|
| A, 3.   | B, 2.  |
| Aq, 9.  | Bq, 4. |
| Ac, 27. | Bc, 8. |

duced



duced of them (Aq, Bq) shall be prime to one another. And if the numbers given at first A, B, multiplying the said produced numbers (Aq, Bq,) produce others (Ac, Bc,) the numbers also shall be prime to one another: And so on.

a 27. 7.

b 28. 7.

For because A is prime to B, (a) therefore Aq shall be prime to B, and Aq being prime to B, (a) therefore Aq shall be also prime to Bq. Again, because A is as well prime to B and Bq, as Aq is to the said B and Bq, (b) therefore shall Ax Aq, that is, Ac, be prime to BxBq, that is, to Bc: And so forth of the rest. *W.W.D.*

### PROP. XXX.

8                      5

A ..... B ..... C 13. D ----

*If two numbers AB, BC, be prime the one to the other; then both added together (AC) shall be prime to either of them AB, BC. And if both added together AC be prime to any one of them AB, the numbers also given in the beginning AB, BC, shall be prime to one another.*

a 12. ax. 7.

1. Hyp. For if you would have AC, AB to be composed, let D be the common measure: (a) this shall measure the residue BC: And therefore AB, BC, are not prime to one another; which is against the Hypothesis.

b 10. ax. 7.

2. Hyp. AC, AB, being taken prime to one another let D be the common measure of AB, BC. (b) But seeing that measures the whole AC, therefore AC, AB, are not prime to one another; contrary to the Hypothesis.

### Coroll.

Hence, a number, which being compounded of two is prime to one of them, is also prime to the other.

### PROP. XXXI.

A, 5. B, 8. *Every prime number A is prime to every number B, which it measureth not.*

a 11. def. 7.

For if any common measure doth measure both, A, B, (a) then A will not be a prime number; contrary to the Hypothesis.

### PROP. XXXII.

A, 4. D, 3.  
B, 6. E, 8.  
AB, 24.

*If two numbers A, B, multiplying another produce another AB, and some prime Number D, measure the number produced of them AB; then shall it measure one of those numbers, A, or B, which were given at the beginning.*

Suppo

Suppose the number D not to measure the number A,  
and let  $\frac{AB}{D}$  be  $\equiv E$ , (a) then  $AB = DE$ ; (b) whence  $D : A :: B : E$ . (c) But D is prime to A; (d) therefore D and A are the least in their proportion; (e) and consequently D measures B as often as A measures E. The proposition therefore is evident.

a 9. ax. 7.  
b 19. 7.  
c hyp. and  
31. 7.  
d 23. 7.  
e 21. 7.

PROP. XXXIII.

Every composed number A, is measured by some prime number B.

A, 12.  
B, 2.

Let one or more numbers (a) measure A, of which let the least be B; that shall be a prime number: For if it be said to be composed, then some (a) lesser number shall measure it, (b) which shall also consequently measure A. Wherefore B is not the least of those which measure A, contrary to the Hyp.

a 13. def. 7.  
b 11. ax. 7.

PROP. XXXIV.

Every number A, is either a prime, or measured by some prime number.

A, 9.

For A is necessarily either a prime or a composed number. If be a prime, 'tis that we affirm. If composed, (a) then some prime number measureth it. Which was to be demonstrated.

a 33. 7.

PROP. XXXV.

A, 6. B, 4. C, 8.

D, 2.

H--I--K----

E, 3. F, 2. G, 4.

L---

How many numbers soever A, B, C, being given to find the least numbers E, F, G, that have the same proportion with them.

If A, B, C, be prime to one another, (a) they shall be the least in their proportion. If they be composed, (b) let their greatest common measure be D, which let measure them by E, F, G. These are then the least in the proportion A, B, C.

a 23. 7.

b 3. 7.

For  $D \times E, F, G$ , (c) produceth A, B, C, (d) therefore these and those are in the same proportion. But allow other numbers H, I, K, to be the least in the same proportion

c 9. ax. 7.  
d 17. 7.

- e 21. 7. portion ; (e) which shall therefore equally measure A, B,  
 f 9. ax. 7. C, namely by the number L, (f) therefore  $L \times H, I, K,$   
 g 1. ax. 1. shall produce A, B, C, (g) and consequently  $ED = A =$   
 h 19. 7. HL ; (h) from whence  $E : H :: L : D$ . But  $E(k) \sqsubset$   
 k suppos. H ; (l) therefore  $L \sqsubset D$ , and so D is not the greatest  
 l 20. def. 7. common measure of A, B, C. Which is against the Hy-  
 pothesis.

Coroll.

Hence, The greatest common measure of how many numbers soever, measures them by the numbers which are least of all that have the same proportion with them. Whence is derived the vulgar method of reducing fractions to their least terms.

## P R O P. XXXVI.

Two numbers being given, A, B, to find out the least number which they measure.

A, 5. B, 4.

AB, 20.

D ----

E --- F ---

1. Case. If A and B be prime the one to the other, AB is the number required. For it is manifest that A and B measure AB. If it be possible, let A and B measure some other number D  $\sqsupset$  AB, suppose by E, and F ; (a) therefore  $AE = D = BF$ , (b) and so  $A : B :: F : E$ . But because A and B are prime one to the other, (d) and so least in their proportion, A shall (e) equally measure F as B does E. But  $B : E(f) :: AB : AE(D)$ . (g) Therefore AB shall also measure D, which is less than it self. Which is absurd.

a 9. ax. 7.

b 1. ax. 1.

b 19. 7.

c hyp.

d 23. 7.

e 21. 7.

f 17. 7.

g 20. def. 7.

A, 6. B, 4. F ----

C, 3. D, 2. G --- H ---

AD, 12.

h 35. 7.

k 19. 7.

l 7. ax. 7.

m 9. ax. 7.

n 19. 7.

o constr.

p 21. 7.

q 17. 7.

r 20. def. 7.

2. Case. But if A and B be composed one to another, (b) let there be found C and D the least in the same proportion. (k) Therefore  $AD = BC$ ; and AD or BC shall be the number sought.

For it is (l) plain that B and A measure AD or BC. Conceive A and B to measure F  $\sqsupset$  AD, namely A by G, and B by H, (m) therefore  $AG = F = BH$ ; whence  $A : B :: H : G(o) :: C : D$ , (p) and consequently C equally measures H as D does G. But  $D : G :: A D : A G(F)$ , therefore AD (r) measures F, the greater the less. Which is absurd.



*Coroll.*

Hence, If two numbers multiply the least that are in the same proportion, the greater the less, and the less the greater, the least number which they measure shall be produced.

P R O P. XXXVII.

If two numbers A, B, measure any number CD, the least number which they measure, E, shall also measure the same CD.

A, 2. B, 3.  
E, . . . 6.  
C - - - F - - - D.

If you deny it, take E from CD as often as you can, and leave FD  $\neg$  E, therefore seeing A and B (a) measure E, (b) and E measures CF, (c) likewise A and B will measure CF. But (a) they measure the whole CD; (d) therefore also they measure the residue FD; and consequently E is not the least which A and B measure: *Contrary to the Hypothesis*.

a hyp.  
b constr.  
c 11. ax. 7.  
d 12. ax. 7.

P R O P. XXXVIII.

Three numbers being given, A, B, C, to find out the least which they measure.

A, 3. B, 4. C, 6.  
D, 12.

(a) Find D the least that two of them A and B measure; which if the third C also measure, it is manifest that D is the number sought. But if C doth not measure D, let E be the least that C and D measure, E shall be the number required.

a 36. 7.

For it appears by the 11. ax. 7. that A, B, C, measure E; and it is easily shewn that they measure no other F less than E.

A, 2. B, 3. C, 4.  
D, 6. E, 12.  
F - - -

For if you affirm they do, (b) then D measures F, (b) and consequently E measures the same F, the greater the less. *Which is absurd*.

b 37. 7.

*Coroll.*

Hence it appears, that if three numbers measure any number, the least also, which they measure, shall measure the same.

H

P R O P.

## The seventh Book of

## P R O P. XXXIX.

A, 12. If any number B, measures a number A,  
B, 4. C, 3. the number measured A, shall have a part  
C denominated of the number measuring B.

a hyp. For because  $\frac{A}{B} (a) = C, (b)$  therefore  $A = BC (c)$  and  
b 9. ax. 7.  $\frac{A}{C} = B$ . Which was to be demonstrated.  
c 7. ax. 7.

## P R O P. XL.

A, 15. If a number A, have any part whatsoever  
B, 3. C, 5. B, the number C, from which the part  
B is denominated, shall measure the same.

a hyp. & 9. For since  $BC (a) = A, (b)$  therefore  $\frac{A}{C} = B$ . Which  
ax. 7. was to be demonstrated.  
b 7. ax. 7.

## P R O P. XLI.

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$  G, 12. To find out a number G, the least that can  
H - - - have given parts,  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ .

a 38. 7. (a) Let G be found the least which the denominators  
b 39. 7. 2, 3, 4, measure; (b) it is evident that G has the parts  
c 40. 7.  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ . If it be possible let H  $\sqsubset$  G have the same  
parts; (c) therefore 2, 3, 4, measure H; and so G is  
not the least which 2, 3, 4, measure: against the supposition.

The End of the seventh Book.

The

# The EIGHTH BOOK OF EUCLIDE'S ELEMENTS.

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## PROP. I.

A, 8. B, 12. C, 18. D, 27.

E-F-G-H----

**I**F there be divers numbers how many soever in continual proportion, A, B, C, D, and their extremes A, D, prime to one another; then those numbers A, B, C, D, are the least of all numbers that have the same proportion with them.

For, if it be possible, let there be as many others E, F, G, H, less than A, B, C, D, and in the same proportion with them. (a) Therefore from equality  $A : D :: E : H$ , and so A and D, which are prime numbers, (b) and consequently the least in their proportion, (c) equally measure E and H, which are less than themselves. Which is absurd.

a 14. 7.  
b 23. 7.  
c 21. 7.

## PROP. II.

A, 2. B, 3.

Aq, 4. AB, 6. Bq, 9.

Ac, 8. AqB, 12. BqA, 18. Bc, 27.

To find out the least numbers continually proportional, as many as shall be required, in the proportion given of A to B.

Let A and B be the least in the proportion given; Then Aq, AB, Bq, shall be the three least in the same continual proportion that A is to B.

For  $AA : AB$  (a)  $:: A : B$  (b)  $:: AB : BB$ . Likewise because A and B are prime one to the other, (c) shall Aq, Bq, be also prime to one another, (d) and so Aq, AB, Bq, are  $::$  the least in the proportion of A to B. More-

a 17. 7.  
b 24. 7.  
c 29. 7.  
d 1, 8.



e 17. 7.

f 29. 7.

g 1. 8.

Moreover, I say  $Ac, AqB, BqA, BC$ , are the four least in the proportion of  $A$  to  $B$ . For  $AqA : AqB(e) :: A : B(e) :: ABA(AqB) : ABB, (e)$  and  $A : B :: ABqA : BBB(Bc)$ . Therefore since  $Ac$ , and  $Bc$ , are  $(f)$  prime to one another, likewise  $(g)$  shall  $Ac, AqB, BqA, Bc$  be the four least  $∴$  in the proportion of  $A$  to  $B$ . In the same manner may you find out as many proportional numbers as you please. *Which was to be done.*

## Coroll.

1. Hence, If three numbers, being the least, are proportional, their extremes shall be squares; if four, cubes.

2. The extremes of any number of proportionals found by this proposition, if such proportionals are the least of all in a given ratio, are prime to one another.

3. Two numbers, being the least in a given ratio, measure all the mean numbers of proportionals be they ever so many, provided they are the least in the same proportion; because they arise from the multiplication of them into certain other numbers.

4. Hence it also appears by the construction, that the series of numbers  $1, A, Aq, Ac$ ;  $1, B, Bq, Bc$ ;  $Ac, AqB, BqA, Bc$ , consist of an equal multitude of numbers; and consequently, the extreme numbers of how many soever the least continually proportionals are the last of as many other continually proportionals from unity; thus the extremes  $Ac, Bc$ , of the continual proportionals  $Ac, AqB, BqA, Bc$ , are the last of as many proportionals from unity  $1, A, Aq, Ac$ , and  $1, B, Bq, Bc$ .

5.  $1, A, Aq, Ac$ ; and  $B, BA, AqB$ ; and  $Bq, BqA$  are  $∴$  in the ratio of  $1$  to  $A$ . Also  $B, Bq, Bc$ ; and  $A, AB, BqA$ ; and  $Aq, AqB$  are  $∴$  in the ratio of  $1$  to  $B$ .

## P R O P. III.

$A, 8. B, 18. C, 12. D, 27.$  *If there be numbers continually proportional, how many soever,  $A, B, C, D$ , being also the least of all that have the same proportion with them; their extremes  $A, D$ , are prime to one another.*

a 2. 8.

For if there be  $(a)$  found as many numbers the least in the proportion of  $A$  to  $B$ , they shall be no other than  $A, B, C, D$ ; therefore, by the second Coroll. of the precedent prop. the extremes  $A$  and  $D$  are prime to one another. *Which was to be demonstrated.*

## P R O P.

PROP. IV.

Proportions how many  
soever being given in the  
least numbers (A, to B,  
and C to D) to find out the  
least numbers continually proportional in the proportions given.

A, 6. B, 5. C, 4. D, 3.  
H, 4. F, 24. E, 20. G, 15  
I -- K -- L --

(a) Find out E, the least number which B and C measure; and let B measure E (b) as often as A does another F, viz. by the same number H. (b) Also let C measure the said E as often as D measures another G, then F, E, G, shall be the least in the proportions given. For AH (c) = F, and BH (c) = E; (d) therefore A : B :: AH : BH (e) :: F : E. In like manner C : D :: E : G; therefore F, E, G, are continually proportional in the proportions given. And they are moreover the least in the said proportions; for conceive other numbers I, K, L, to be the least; (f) then A and B must equally measure I and K, (f) and C and D likewise K and L; and so B and C measure the same K. (g) Wherefore also E measures the same number K, which is less then it self. Which is absurd.

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.  
H, 24. G, 20. I, 15. K, 21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the least continually in the proportions of A to B, and C to D. Then if E measures I, (b) take another number K, which may be equally measured by F; and those four numbers H, G, I, K, shall be continually the least in the given proportions; which we need go no other way to prove than we did in the first part.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.  
H, 24. G, 20. I, 15.  
M, 48. L, 40. K, 30. N, 105.

If E doth not measure I, let K be the least which E and I measure; and as often as I measures K, let G as often measure L, and H also M, so likewise let F measure N as often as E measures K. The four numbers M, L, K, N, shall be the least continually in the given proportions; which we may demonstrate as before.

a 36. 7.  
b 3 post. 7.

c 9. ax. 7.  
d 18. 7.  
e 7. 5.

f 21. 7.

g 37. 7.

h 3 post. 7.

## P R O P. V.

C, 4. E, 3.  
D, 6. F, 16. ED, 18.  
CD, 24. EF, 48.

Plane numbers CD, EF,  
are in that proportion to one  
another which is composed  
of their sides.

a 17. 7.  
b 20. def. 5.  
c 11. 5.

For because  $CD : DE (a) :: C : E$ ;  $(a)$  and  $ED : EF$   
 $:: D : F$  and  $\frac{CD}{EF} (b) = \frac{CD}{ED} \times \frac{ED}{EF} (c)$  then shall be the  
proportion  $\frac{CD}{EF} = \frac{C}{E} \times \frac{D}{F}$ . Which was to be demon.

## P R O P. VI.

A, 16. B, 24. C, 36. D, 54. E, 81.  
F, 4. G, 6. H, 9.

If there be num-  
bers continually pro-  
portional how many

soever, A, B, C, D, E, and the first A does not measure the  
second B, neither shall any of the other measure any one of  
the rest.

a 20. def. 7.

b 35. 7.

c 5. 7.

d 3. 3.

e 14. 7.

Since A does not measure B,  $(a)$  neither shall any one  
measure that which next follows; Because  $A : B :: B : C$   
 $:: C : D$ , &c.  $(b)$  Take three numbers, F, G, H, the least  
in the proportion of A to B, therefore since A does not  
measure B,  $(a)$  neither shall F measure G,  $(c)$  therefore  
F is not unity. But F and H are prime one to another;  
therefore  $(d)$  since by equality  $A : C :: F : H$ , and F  
does not measure H,  $(a)$  neither shall A measure C; and  
consequently neither shall B measure D, nor C measure  
E, &c. because  $A : C (e) :: B : D (e) :: C : E$ , &c.  
In like manner four or five numbers being taken the  
least in the proportion of A to B, it may be shewn that  
A does not measure D and E; nor does B measure E and  
F, &c. Wherefore none of them shall measure any other.  
Which was to be demonstrated.

## P R O P. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

If there be numbers continually proportional how many so-  
ever A, B, C, D, E, and the first A measures the last E, it  
shall also measure the second B.

a 6. 7.

If you deny that A measures B,  $(a)$  then neither shall  
it measure E; Which is contrary to the Hyp.

P R O P.



PROP. VIII.

If between two numbers A, B, there fall mean numbers in continual proportion C, D; as many mean numbers in continual proportion as fall between them, so many mean numbers also L, M, in continual proportion, shall fall between two other numbers E, F, which have the same proportion with them (L, M,)

(a) Take G, H, I, K, the least  $\div$  in the proportion of A to C; (b) by equality it shall be  $G : K :: A : B$  (c)  $E : F :: G : K$ . But G, and K (d) are prime one to another. (e) Wherefore G measures E as often as K does F. Let H measure L, and I likewise M by the same number; (f) therefore E, L, M, F, are in such proportion as G, H, I, K, that is, as A, B, C, D. Which was to be demonstrated.

a 35. 7.  
b 14. 7.  
c hyp.  
d 3. 8.  
e 21. 7.  
f constr

PROP. IX.

If two numbers A, B are prime to one another, and mean numbers in continual proportion C, D, fall between them; as many mean numbers in continual proportion as fall between them, so many means also in continual proportion (E, G; and F, I) shall fall between either of them and unity.

It is evident, that 1, E, G, A, and 1, F, I, B, are  $\div$ , and as many as A, C, D, B, namely by the 4th Coroll. 2. 8. Which was to be demonstrated.

PROP. X.

If between two numbers A, B, and an unit, numbers continually proportional (E, D, and F, G,) fall as many mean numbers in continual proportion as fall between either of them and unity, so many means also shall fall in continual proportion between them, I, K.

For E, DF, G, and A, DqF (I) DG (K) B are  $\div$  by 2. 8, therefore, &c.

## P R O P. XI.

A, 2. B, 3.  
Aq, 4. AB, 6. Bq, 9. *Between two square numbers Aq, Bq, there is one mean proportional number AB; and Aq to Bq, is in duplicate proportion of the side A to the side B.*

a 17. 7.  
b 10. def. 5.

(a) It is manifest that Aq, AB, Bq, are  $\therefore$ ; (b) and consequently also  $\frac{Aq}{Bq} = \frac{A}{B}$  twice. Which was to be demonstrated.

## P R O P. XII.

Ac, 27. AqB, 36. BqA, 48. Bc, 64.  
A, 3. B, 4.  
Aq, 9. AB, 12. Bq, 16. *Between two cube numbers, Ac, Bc, there are two mean proportional numbers AqB, BqA; and the cube Ac is to the cube Bc in triplicate ratio of the side A to the side B.*

a 2. 8.

b 10. def. 5. (a) For Ac, AqB, BqA, Bc, are  $\therefore$  in the proportion of A to B; (b) and therefore  $\frac{Ac}{Bc} = \frac{A}{B}$  thrice. Which was to be demonstrated.

## P R O P. XIII.

A, 2. B, 4. C, 8.  
Aq, 4. AB, 8. Bq, 16. BC, 32. Cq, 64.  
AC, 8. AqB, 16. BqA, 32. Bc, 64. BqC, 128. CqB, 256. Cc, 512.

*If there be numbers in continual proportion how many soever A, B, C; and every of them multiplying itself produceth certain numbers; the numbers produced of them Aq, Bq, Cq, shall be proportional: And if the numbers first given A, B, C, multiplying their products Aq, Bq, Cq, produce other numbers Ac, Bc, Cc, they also shall be proportional; and this shall ever happen to the extremes.*

For Aq, AB, Bq, BC, Cq (a) are  $\therefore$ ; (b) therefore by equality Aq : Bq :: Bq : Cq. Which was to be demonstrated.

a 2. 8.

b 14. 7i

(a) Also Ac, AqB, BqA, Bc, BqC, CqB, Cc, are  $\therefore$ ; (b) therefore again by equality Ac : Bc :: Bc : Cc. Which was to be demonstrated.

## P R O P.

P R O P. XIV.

If a square number Aq mea- Aq. 4. AB, 12. Bq, 36.  
sure a square number Bq, the side A, 2. B, 6.  
also of the one (A) shall measure  
the side of the other (B) : and if the side of one square A,  
measure the side of another B; the square Aq shall likewise  
measure the square Bq.

1. Hyp. For Aq : AB (a) :: AB : Bq, therefore seeing by  
the hypothesis Aq measures Bq, (b) it shall measure also  
AB. But Aq : AB :: A : B, (c) therefore also A measures  
B. Which was to be demonstrated.

2. Hyp. A measures B; (c) therefore Aq shall as well  
measure AB, (c) as AB measures Bq; (d) consequently Aq  
measures Bq. Which was to be demonstrated.

a 2. & 11. 8

b 7. 8.

c 20. def. 7.

d 11. ax. 7.

P R O P. XV.

If a cube number A, 2. B, 6.  
Ac measures a cube Ac, 8. AqB, 24. ABq, 72. Bc, 216.  
number Bc, then the  
side of the one (A) shall measure the side of the other (B).  
And if the side A of one cube Ac measure the side B of the  
other Bc, also the cube Ac shall measure the cube Bc.

1. Hyp. For Ac, AqB, BqA, Bc (a) are ::; therefore  
Ac, (b) measuring the extreme Bc, shall also (c) measure the  
second AqB. But Ac : AqB :: A : B, (d) therefore A  
shall also measure B. Which was to be dem.

2. Hyp. A measures B; (d) therefore Ac measures  
AqB, which also measures BqA, and this Bc; (e) there-  
fore Ac shall measure Bc. Which was to be dem.

a 2. & 12. 8

b hyp.

c 7. 7.

d 20. def. 7

e 11. ax. 7.

P R O P. XVI.

If a square number Aq do not mea- A, 4. B, 9.  
sure a square number Bq, neither shall Aq, 16. Bq, 81.  
the side of the one A measure the side  
of the other B : And if the side of the one square Aq do not  
measure B the side of the other Bq, neither shall the square  
Aq measure the square Bq.

1. Hyp. For if you affirm that A measures B, then  
Aq also shall measure Bq. Against the Hypothesis.

2. Hyp. If you maintain Aq to measure Bq; (a) then  
likewise A shall measure B. Contrary to the Hypothesis.

a 14. 8.

P R O P.



## The eighth Book of

## P R O P. XVII.

A, 2. B, 3. If a cube number Ac does not measure a cube number Bc, neither shall the side of one A measure the side of the other B: And if A the side of one cube Ac does not measure B the side of the other Bc, neither shall the cube Ac measure the cube Bc.

a 15. 8.

1. Hyp. Let A measure B; (a) then Ac shall measure Bc. Against the Hypothesis.

2. Hyp. Let Ac measure Bc; then A shall measure B; which is also against the Hypothesis.

## P R O P. XVIII.

C, 6. D, 2.

CD, 12.

E, 9. F, 3. DE, 18.

EF, 27.

Between two like plane numbers CD and EF there is one mean proportional number DE: And the plane CD is to the plane EF in duplicate proportion of that

which the side C hath to the homologous side E.

\* 21. def. 7.

a 17. 7.

b 11. 5.

c 10. def. 5.

For \* by the Hypothesis  $C : D :: E : F$ ; therefore by permutation,  $C : E :: D : F$ . But  $C : E (a) :: CD : DE$ ; (a) and  $D : F :: DE : EF$ ; (b) therefore  $CD : DE :: DE : EF$ . (c) Wherefore the proportion of CD to EF is duplicate to that of CD to DE, that is, to the proportion of C to E, or D to F. Which was to be demonstrated.

Coroll.

Hence it is apparent, That between two like plane numbers there falls one mean proportional in the proportion of the homologous sides.

## P R O P. XIX.

CDE, 30. DEF, 60. FGE, 120. FGH, 240.

CD, 6. DF, 12. FG, 24.

C, 2. D, 3. E, 5. F, 4. G, 6. H, 10.

Between two like solid numbers CDE, FGH, there are two mean proportional numbers DFE, FGE. And the solid CDE is to the solid FGH, in triplicate proportion of that which the homologous side C has to the homologous side F.

Where-

Whereas by the \* hyp.  $C : D :: F : G$ , and  $D : E :: G : H$ ; therefore (a) by permutation, shall  $C : F :: D : G$ ; (a)  $E : H$ . But  $CD : DF$  (b)  $:: C : F$ , and  $DF : FG$  (b)  $:: D : G$ ; (c) wherefore  $CD : DF :: DF : FG :: E : H$ ; (d) therefore  $CDE : DFE :: DFE : FGE :: E : H$ ;  $FGE : FGH$ . Therefore between  $CDE$ ,  $FGH$ , fall two mean proportionals  $DFE$ ,  $FGE$ . (e) And so it is plain that the proportion of  $CDE$  to  $FGH$  is triplicate to that of  $CDE$  to  $DFE$ , or  $C$  to  $F$ . Which was to be dem.

\* 21. def. 7.  
a 13. 7.  
b 17. 7.  
c 11. 5.  
d 17. 7.  
e 10. def. 5.

Coroll.

Hereby it is manifest, that between two like solid numbers there fall two mean proportionals in the proportion of the homologous sides.

P R O P. XX.

If between two numbers  $A$ ,  $A, 12.$   $C, 18.$   $B, 27.$   
 $B$ , there falls one mean proportional number  $C$ ; those numbers  $A, B$ , are like plane numbers.  
 $D, 2.$   $E, 3.$   $F, 6.$   $G, 9.$

(a) Take  $D$  and  $E$  the least in the proportion of  $A$  to  $C$ , or  $C$  to  $B$ , then  $D$  measures  $A$  equally as  $E$  does  $C$ ; suppose by the same number  $F$ ; (b) also  $D$  equally measures  $C$ , as  $E$  does  $B$ , suppose by the same number  $G$ . (c) Therefore  $DF = A$ , and  $EG = B$ ; (d) and consequently  $A$  and  $B$  are plane numbers. But because  $EF$  (c)  $= C$  (c)  $= DG$ ; (e) therefore  $D : E :: F : G$ , and alternately  $D : F :: E : G$ . (f) Therefore the plane numbers  $A$  and  $B$  are also like. Which was to be dem.

a 35. 7.  
b 21. 7.  
c 9. ax. 7.  
d 16. def. 7.  
e 19. 7.  
f 21. def. 7.

P R O P. XXI.

If between two numbers  $A, 16.$   $C, 24.$   $D, 36.$   $B, 54.$   
 $A, B$ , there fall two mean proportional numbers  $C, D$ ;  $E, 4.$   $F, 6.$   $G, 9.$   
those numbers  $A, B$ , are like solid numbers.  $H, 2.$   $P, 2.$   $M, 4.$   $K, 3.$   $L, 3.$   $N, 6.$

(a) Take  $E, F, G$ , the least in the proportion of  $A$  to  $C$ , (b) then  $D$  and  $G$  are like plane numbers: let the sides of this be  $H$  and  $P$ ; and of that  $K$  and  $L$ ; (c) therefore  $H : K :: P : L$ ; (d)  $E : F$ . But  $E, F, G$ , (e) equally measure  $A, C, D$ , suppose by the same number  $M$ , and likewise the said numbers  $E, F, G$ , equally measure the numbers  $C, D, B$ , suppose by the same number,  $N$ . (f) Therefore  $A = EM = HPM$ , (f) and  $B = GN = KLN$ ; (g) and

a 2. 8.  
b 10. 8.  
c 21. def. 7.  
d cor. 18. 8.  
e 21. 7.  
f 9. ax. 7.

- g 17. *def.* 7. (g) and so A and B are solid numbers. But because C (f)  
 h 17. 7. = FM, and D (f) = FN, therefore M : N (b) : : FM : FN  
 k 7. 5. FN (k) : : C : D (l) : : E : F : : H : K : : P : L ; (m)  
 l *const.* therefore A and B are like solid numbers. Which was to  
 m 21. *def.* 7. be demonstrated.

*Lemma.*

AE, BF, CG, DH, If proportional numbers A, B  
 A, B, C, D, C, D, measure proportional num-  
 E, F, G, H, bers AE, BF, CG, DH, by the  
 numbers E, F, G, H, these num-  
 bers (E, F, G, H,) shall be proportional.

- a 19. 7. For because AEDH (a) = BFCG, (a) and AD = EC  
 b 1. *ax.* 7. (b) shall  $\frac{AEDH}{AD} = \frac{BFCG}{BC}$  (c) that is, E H = F G  
 c 9. *ax.* 7. (a) Therefore E : F : : G : H. Which was to be demon-

*Coroll.*

- d 15. *def.* 7. Hence  $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$  (e) For 1 : B : : B : Bq, (d) and  
 e *lem. prec.* 1 : A : : A : Aq, (d) therefore 1 :  $\frac{B}{A}$  : :  $\frac{B}{A}$  :  $\frac{Bq}{Aq}$  (d) there-  
 fore  $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$ . In like manner  $\frac{B}{Ac} \times \frac{Bq}{Ac} = \frac{Bc}{Acc}$   
 and so of the rest.

### P R O P. XXII.

Aq, B, C. If three numbers Aq, B, C, are contin-  
 4, 8, 16, ally proportional, and the first Aq a square,  
 the third C shall also be a square.

- a 20. 7. For because AqC (a) = Bq, (b) thence is C =  $\frac{Bq}{Aq}$   
 b 7. *ax.* 7. (c) =  $Q \cdot \frac{B}{A}$ . But it is plain that  $\frac{B}{A}$  is a number, (d) be-  
 c *cor. of the* cause  $\frac{Bq}{Aq}$  or C is a number. Therefore if three, &c.  
*lem. prec.*  
 d *hyp. and*  
 14. 8.

### P R O P. XXIII.

Ac, B, C, D, If four numbers A, B, C, D, are  
 8, 12, 18, 27. continually proportional, and the first  
 them Ac a cube, the fourth also  
 all be a cube.



For because  $Ac D (a) = BC$ ,  $(b)$  therefore  $D = \frac{BC}{Ac}$

$(c) = \frac{B}{Ac} \times C$ ; that is, (because  $Ac C = (d) Bq$ , and

$(b)$  thence  $C = \frac{Bq}{Ac}$ )  $D = \frac{B}{Ac} \times \frac{Bq}{Ac} (c) = \frac{Bc}{Acc} (c) = C \cdot \frac{B}{Ac}$

But it is evident  $(e)$  that  $\frac{B}{Ac}$  is a number, because  $\frac{Bc}{Acc}$

or  $D$  is supposed a number. Therefore if four numbers,

PROP. XXIV.

If two numbers  $A, B$ , are in the same proportion one to another, that a square number  $C$  is to a square number  $D$ , and the first  $A$  be a square number, the second also  $B$  shall be a square number.

Between  $C$  and  $D$  the square numbers, \* and so between  $A$  and  $B$  having the same proportion,  $(a)$  falls one mean proportional. Therefore  $(b)$  since  $A$  is a square number,  $(c)$   $B$  also shall be a square number. Which was to be demonstrated.

Coroll.

1. Hence, if there be two like numbers  $AB, CD$ , ( $A : B :: C : D$ ) and the first  $AB$  be a square, the second also  $CD$  shall be a square.

\* For  $AB : CD :: Aq : Cq$ .

2. From hence it appears, That the proportion of any square number to any other not square, cannot possibly be resolved into two square numbers. Whence it cannot be  $Q : Q :: 1 : 2$ , nor  $1 : 5 :: Q : Q$ , &c.

PROP. XXV.

If two numbers  $A, B$ , are in the same proportion one to another, that a cube number  $C$  is to a cube number  $D$ , the first of them  $A$  being a cube number; the second  $B$  shall likewise be a cube number.

$(a)$  Between the cube numbers  $C$  and  $D$ ,  $(b)$  and so between  $A$  and  $B$  having the same proportion, fall two mean proportionals; therefore  $(c)$  because  $A$  is a cube,  $(d)$  shall  $B$  be a cube also. Which was to be demonstrated.

Coroll.

a 19. 7.  
b 7. ax 7.  
c cor. of the  
prec. lem.  
d 20. 7.

e 15. 8.

\* 8. 8.  
a 11. 8.  
b hyp.  
c 22. 8.

\* 11 and  
18. 8.

a 12. 8.  
b 8. 8.  
c hyp.  
d 23. 8.

## Coroll.

1. Hence, If there be two numbers ABC, DEF, ( $A : B :: D : E$ , and  $B : C :: E : F$ ;) and the first ABC be a cube, the second DEF shall be a cube also.

\* 12. and  
19. 8.

\* For  $ABC : DFF :: Ac : Dc$ .

2. Hence it is evident, That the proportion of any cube number to any other number not a cube cannot be found in two cube numbers.

## P R O P. XXVI.

A, 20. C, 30. B, 45.  
D, 4. E, 6. F, 9.

Like plane numbers A, B,  
are in the same proportion one  
to another, that a square num-

ber is in to a square number.

a 18. 8.  
b 2. 8.  
c cor. 2. 8.  
d 14. 7.

Between A and B (*a*) falls one mean proportional number C; (*b*) take three numbers D, E, F, the least  $::$  in the proportion of A to C, the extremes D, F, (*c*) shall be square numbers. But by equality  $A : B (d) :: D : F$ ; therefore  $A : B :: Q : Q$ . Which was to be demonstrated.

## P R O P. XXVII.

A, 16. C, 24. D, 36. B, 54.  
E, 8. F, 12. G, 18. H, 27.

Like solid numbers  
A, B, are in the  
same proportion one

to another, that a cube number is to a cube number.

a 19. 8.  
b 2. 8.

c 14. 7.

(*a*) Between A and B fall two mean proportional numbers, namely, C and D: (*b*) Take four numbers E, F, G, H, the least  $::$  in the proportion of A to C, (*c*) the extremes E, H, are cube numbers. But  $A : B (c) :: E : H :: C : C$ . Which was to be demonstrated.

## Schol.

See Clavius.

1. From hence is inferred, that no numbers in proportion superparticular, or superbipartitent, or double, or any other manifold proportion not denominated from a square number, are like plane numbers.

2. Likewise, that neither any two prime numbers, nor any two numbers prime one to another, not being squares, can be like plane numbers.

The End of the eighth Book.

THE

## The NINTH BOOK

O F

E U C L I D E ' S  
E L E M E N T S .

## PROPOSITION I.

A, 6. B, 54.

Aq, 36. 108. AB, 324.

**I** F two like plane numbers A, B, multiplying one another, produce a number AB, the number produced AB shall be a square number.

For  $A : B (a) :: Aq : AB$ ; wherefore since one mean proportional ( $b$ ) falls between A and B, ( $c$ ) likewise one mean proportional number shall fall between Aq and AB: therefore since the first Aq is a square number, ( $d$ ) the third AB shall be a square number also. Which was to be demonstrated.

Or thus, Let  $ab, cd$ , be like plane numbers; namely,  $a : b :: c : d$ , ( $x$ ) therefore  $ad = bc$ , and so likewise  $abcd$  or  $adbc$ , ( $y$ )  $= adad = Q : ad$

a 17. 7.

b 18. 8.

c 8. 8.

d 22. 8.

x 19. 7.

y 1. ar. 7.

## PROP. II.

If two numbers A, B, multiplying one another, produce a square number AB, those numbers A, B, are like plane numbers.

A, 6. B, 54.

Aq, 36. AB, 324.

For  $A : B (a) :: Aq : AB$ ; wherefore since between Aq and AB, ( $b$ ) there falls one mean proportional number, ( $c$ ) likewise one mean shall fall between A and B, ( $d$ ) therefore A and B are like planes. Which was to be demonstrated.

a 17. 7.

b 11. 8.

c 8. 8.

d 20. 8.

## P R O P .



## P R O P. III.

A, 2. Ac, 8. Acc, 64.

If a cube number Ac multiplied by it self produce a number Acc, the number produced Acc shall be a cube number.

a 15. def. 7.

b 17. 7.

c 8. 8.

d 23. 8.

For,  $1 : A(a) :: A : Aq(b) :: Aq : Ac$ , therefore between 1 and Ac fall two mean proportionals. But  $1 : A(a) :: Ac : Acc$ ; (c) therefore between Ac and Acc, fall also two mean proportionals; and so by consequence, since Ac is a cube (d) Acc shall be a cube also. Which was to be demonstrated.

Or thus;  $aaa(Ac)$  multiplied into it self make  $aaaaa(Acc)$ ; this is a cube, whose side is  $aa$ .

## P R O P. IV.

Ac, 8. Bc, 27.  
Acc, 64. AcBc, 216.

If a cube number Ac multiplied by a cube number Bc produce a number AcBc, the produced number AcBc shall be a cube.

a 17. 7.

b 12. 8.

c 8. 8.

d 23. 8.

For  $Ac : Bc(a) :: Acc : AcBc$ . But between Ac and Bc (b) two mean proportional numbers fall; (c) therefore there fall as many between Acc and AcBc. So that where as Acc is a cube number, (d) AcBc shall be such also. Which was to be demonstrated.

Or thus.  $AcBc = aaa bbb (ababab) = C : ab$ .

## P R O P. V.

Ac, 8. B, 27.  
Acc, 64. AcB, 216.

If a cube number Ac multiplied by a number B produce a cube number AcB, the number multiplied B shall also be a cube.

a 17. 7.

b 12. 8.

c 8. 8.

d 23. 8.

For  $Acc : AcB(a) :: Ac : B$ . But between Acc and AcB (b) fall two mean proportionals, (c) therefore also as many shall fall between Ac and B, whence Ac being a cube number, (d) B shall be a cube number also. Which was to be demonstrated.

PROP. VI.

If a number A multiplying  
it self produce Aq a cube, that  
number A it self is a cube.

For because Aq (a) is a cube, and AqA (Ac)(b) also a  
cube; therefore (c) shall A be a cube. Which was to  
be demonstrated.

a hyp.  
b 19. def.  
7.  
c 5. 9.

PROP. VII.

If a composed number A mul-  
tiplying any number B, produce a  
number AB, the number produced  
AB shall be a solid number.

Since A is a composed number, (a) some other number  
D measures it, conceive by E, (b) therefore  $A = DE$  :  
(c) whence  $DEB = AB$  is a solid number, Which was to  
be demonstrated.

a 13. def. 7  
b 9. ax. 7,  
c 17. def.

PROP. VIII.

1. a, 3.  $a^2$ , 9.  $a^3$ , 27.  $a^4$ , 81.  $a^5$ , 242.  $a^6$ , 729.

If from unity there are numbers continually proportional  
how many soever (1, a,  $a^2$ ,  $a^3$ ,  $a^4$ , &c.) the third number from  
unity  $a^2$  is a square number; and so are all forward,  
leaving one between ( $a^4$ ,  $a^6$ ,  $a^8$ , &c.) But the fourth  $a^3$   
is a cube number; and so are all forward, leaving two be-  
tween ( $a^6$ ,  $a^9$ , &c.) The seventh also  $a^6$  is both a cube  
number and a square; and so are all forward, leaving five  
between ( $a^{12}$ ,  $a^{18}$ , &c.)

For 1.  $a^2 = Q : a$ , and  $a^4 = aaaa = Q : aa$ , and  $a^6 =$   
 $aaaaaa = Q : aaa$ , &c.

2.  $a^3 = aaa = C : a$ , and  $a^6 = aaaaaa = C : aa$ , and  
 $aaaaaaaa = C : aaa$ , &c.

3.  $a^6 = aaaaaa = C : aa = Q : aaa$ , therefore, &c.

Or according to Euclide; Because  $1 : a (a) :: a : a^2$ , (b)  
shall  $a^2 = Q : a$ , therefore seeing  $a^2$ ,  $a^3$ ,  $a^4$ , are  $::$  (c) the  
third  $a^4$  shall be a square number; and so likewise  $a^6$ ,  
 $a^9$ , &c. Also because  $1 : a (a) :: a^2 : a^3$ , therefore shall  
 $a^3$ , (b)  $= a^2 \times a = C : a$ , (d) therefore the fourth from  $a^3$ ,  
namely  $a^6$ , shall be likewise a cube, &c. and consequent-  
ly  $a^6$  is both a cube and a square number, &c.

a hyp.  
b 20. 7.  
c 12. 8.  
d 23. 8.

## PROP. IX.

1.  $a, 4. a^2, 16. a^3, 64. a^4, 256, \&c.$ 1.  $a, 8. a^2, 64. a^3, 512. a^4, 4096.$ 

If from unity  
there are numbers  
how many soever,

continually proportional (1.  $a^2, a^3, \&c.$ ) and the number following unity, ( $a$ ) be a square; then all the rest,  $a^2, a^3, a^4, \&c.$  shall be squares also. But if the number next unity ( $a$ ) be a cube, then all the following numbers  $a^2, a^3, a^4, \&c.$  shall be cube numbers.

a 22. 8.

b 23. 8.

c 20. 7.

d 3. 9.

e 23. 8.

1. Hyp. For  $a^2, a^4, a^6, \&c.$  are square numbers by the preceding prop. also since  $a$  is taken to be a square, ( $a$ ) therefore the third  $a^3$  shall be a square, and likewise  $a^5, a^7, \&c.$  and so all.

2. Hyp.  $a$  is put a cube, ( $b$ ) therefore  $a^2, a^3, a^4, \&c.$  are cubes; but by the prec.  $a^2, a^4, a^6, \&c.$  are cubes: lastly, because  $1 : a :: a : aa$ , ( $c$ ) therefore shall  $a^2 = Q : a$ , but a cube multiplied into it self ( $d$ ) produces a cube; therefore  $a^2$  is a cube, ( $e$ ) and consequently the fourth from it  $a^5$ ; and in like manner  $a^8, a^{11}, \&c.$  are cubes, therefore all,  $\&c.$  Which was to be demonstrated.

More clearly thus. Let  $b$  be the side of the square number  $a$ , and so the series  $a, a^2, a^3, a^4, \&c.$  will be otherwise expressed, thus,  $bb, b^2, b^3, b^4, \&c.$  It is evident that all these numbers are squares, and may be thus expressed,  $Q : b, Q : bb : Q : bbb, Q : bbbb, \&c.$

In like manner, if  $b$  be in the side of the cube  $a$ , the series may be expressed thus,  $b^3, b^6, b^9, b^{12}, \&c.$  or  $C : b, C : b^2, C : b^3, C : b^4, \&c.$

## PROP. X.

1,  $a, a^2, a^3, a^4, a^5, a^6$ 

1, 2, 4, 8, 16, 32, 64.

If from unity, there are  
numbers how many soever con-  
tinually proportional (1.  $a, a^2,$

$a^3, \&c.$ ) and the number next the unit ( $a$ ) be not a square number; then is none of the rest following a square number excepting  $a^2$  the third from unity, and so all forward, leaving one between ( $a^4, a^6, a^8, \&c.$ ) But if that ( $a$ ) which is next after the unit, be not a cube number, neither is any other of the following numbers a cube, saving  $a^3$ , the fourth from the unit, and so all forward, leaving two between,  $a^6, a^9, a^{12}, \&c.$

1. Hyp.



1. Hyp. For if it be possible, let  $a^5$  be a square number; therefore because  $a : a^3 (a) :: a^4 : a^5$ , and by inversion,  $a^5 : a^4 :: a^2 : a$ ; and also  $a^5$  and  $a^4 (b)$  square numbers, and the first  $a^2$  a square, (c) therefore  $a$  shall be likewise a square; *contrary to the hyp.*

a hyp.  
b suppos. &  
8. 9.  
c 24. 8.  
d 14. 7.  
e 25. 8.

2. Hyp. If it may be, let  $a^4$  be a cube; since (d) by equality  $a^4 : a^6 :: a : a^3$ , and inversely  $a^6 : a^4 :: a^3 : a$ ; and also since  $a^6$  and  $a^4$  are cubes, and the first  $a^3$  a cube, (e) therefore  $a$  shall be a cube also; *against the hypothesis.*

PROP. XI.

If there are numbers how many soever in continual proportion from unity (1, a,  $a^2$ ,  $a^3$ , &c.) the less measureth the greater by some one of them that are amongst the proportional numbers.

1. a,  $a^2$ ,  $a^3$ ,  $a^4$ ;  $a^5$ ,  $a^6$ ,  
1, 3, 9, 27, 81, 243, 729.

Because  $1 : a :: a : aa (a)$  therefore  $\frac{aa}{a} = a = \frac{aaa}{aa}$  Also

a 5. ax. 7.  
& 20. def. 7.

because  $1 : aa(b) :: a : aaa$ , therefore  $(a) \frac{aaa}{a} = aa =$

b 14. 7.

$\frac{a^4}{aa} = \frac{a^5}{a^3}$ , &c. Lastly because  $1 : a^2 :: (b) a : a^3$ , therefore  $\frac{a^3}{a^2} = \frac{a^6}{a^4}$ , &c.

Coroll.

Hence, If a number that measures any one of proportional numbers, be not one of the said numbers, neither shall the number by which it measures the said proportional numbers, be one of them.

PROP. XII.

1, a,  $a^2$ ,  $a^3$ ,  $a^4$   
1, 6, 36, 216, 1296.  
B, 3.  
If there are numbers how many soever in continual proportion from unity (1, a,  $a^2$ ,  $a^3$ ,  $a^4$ ) whatsoever prime number B, measure the last  $a^4$ , the same (B) shall also measure the number (a) which follow next after unity.

If you say B does not measure a, (a) then B is prime to a; (b) and therefore B is prime to  $a^2$ ; (c) and so consequently to  $a^4$ , which it is supposed to measure. Which is absurd.

a 31. 7.  
b 27. 7.  
c 26. 7.

I 2

Coroll.

unity  
numbers  
soever,  
er fol-  
,  $a^4$ ,  
unity  
, &c.

rs by  
quare,  
ewise

o, are  
ubes:  
Q: a,  
cube;  
fourth  
cubes,

quare  
will  
c. It  
may  
, &c.  
a, the  
or C:

re are  
er con-  
a,  $a^2$ ,  
square  
num-  
ward,  
at (a)  
ither is  
fourth  
n,  $a^4$ ,

Hyp.

## Coroll.

1. Therefore every prime number that measures the last, does also measure all those other numbers that precede the last.

2. If any number not measuring that next to unity, does yet measure the last, it is a composed number.

3. If the number next to the unit be a prime, no other prime number shall measure the last.

## P R O P. XIII.

1,  $a$ ,  $a^2$ ,  $a^3$ ,  $a^4$

1, 5, 25, 125, 625.

H -- G -- F -- E --

If from unity there are numbers in continual proportion, how many soever (1,  $a$ ,  $a^2$ ,  $a^3$ , &c.) and that after unity ( $a$ ) a prime; then shall

no other measure the greatest number, but those which are amongst the said proportional numbers.

If it be possible, let some other E measure  $a^4$ , viz. by  
 a cor. 12. 9. F, ( $a$ ) then F shall be some other different from  $a$ ,  $a^2$ ,  
 b 2. cor. 12. 9.  $a^3$ . But because E measuring  $a^4$ , does not measure  $a$ ,  
 c 33. 7. ( $b$ ) therefore E shall be a composed number, ( $c$ ) therefore  
 d 11. ax. 7. some prime number measures it, ( $d$ ) which does conse-  
 e 3. cor. 12. quently measure  $a^4$ , ( $e$ ) and so is no other than  $a$ , there-  
 f 9. ax. 7. fore  $a$  measures E. After the same manner also may  
 g 19. 7. F be shewn to be a composed number, measuring  $a^4$ ,  
 h 20. def. 7. and so that  $a$  measures F. Therefore seeing  $EF(f) = a^4$ ,  
 k cor. 11 9.  $= a \times a^3$ , ( $g$ ) it will be  $a : E :: F : a^3$ . Consequently  
 whereas,  $a$  measures E, ( $h$ ) likewise F shall equally mea-  
 sure  $a^3$ , viz. by the same number G. ( $k$ ) Nor shall G be  
 $a$ , or  $a^2$ , therefore, as before, G is a composed number,  
 and  $a$  measures it. Wherefore since  $FG(f) = a^3 = a^2 \times a$ ,  
 ( $g$ ) therefore  $a : F :: G : a^2$ , and so because  $A$  measures  
 F, ( $h$ ) G shall equally measure  $a^2$ , viz. by the same num-  
 ber H, ( $k$ ) which is not  $a$ . Therefore since  $GH = a^2 =$   
 $aa(l)$  thence  $H : a :: a : G$ , and because  $a$  measures G  
 l 20. 7. (as before ( $m$ ) H also shall measure  $a$ , which is a prime  
 m 20. def. 7. number. Which is impossible.

P R O P.

PROP. XIV.

If certain prime numbers B, C, D, measure the least number A, no other prime number E shall measure the same besides those that measured it at first.

A, 30.  
B, 2. C, 3. D, 5.  
E - - F - - -

If it is possible, let  $\frac{A}{E}$  be  $= F$ ; (a) then  $A = EF$ , (b)

a 9. ax. 7.  
b 32. 7

therefore each of the prime numbers B, C, D, measures one of those E, F. Not E, which is taken to be a prime; therefore F, which is less than A it self; contrary to the hypothesis.

PROP. XV.

If three numbers continually proportional A, B, C, are the least of all that have the same proportion with them; any two of them added together shall be a prime to the third.

A, 9. B, 12. C, 16.  
D, 3. E, 4.

(a) Take D and E the least in proportion of A to B; (b) then  $A = Dq$ , and (c)  $C = Eq$ , (d) and  $B = DE$ . But because D (c) is prime to E, (d) therefore shall  $D + E$  be prime to both D and E, \* therefore  $D \times \overline{D + E}$  (e)  $= Dq + DE$  (A + B) (f) is prime to E, and so to C or Eq. Which was to be demonstrated.

a 35. 7.  
b 2. 8.  
c 24. 7.  
d 30. 7.  
\* 26. 7.  
e 3. 2,  
f as before  
g 27. 7.

(g) In like manner  $DE + Eq$  (B + C) is prime to D, and consequently to  $A = Dq$ . Which was to be demonstrated.

Lastly, because B (b) is prime to  $D + E$ , it shall also be prime to the square of it (k)  $Dq + 2 DE + Eq$  (A + 2 B + C;) (l) wherefore the said B shall be prime to A + B + C, (l) and so likewise to A + C. Which was to be demonstrated.

h 26. 7.  
k 4. 2.  
l 30. 7.

PROP. XVI.

If two numbers A, B, are prime to one another, it shall not be as the first A, to the second B, so is the second B to any other C.

A, 3. B, 5. C - - -



a 23. 7.  
b 21. 7.  
c 6. ax. 7.

If you affirm  $A : B :: B : C$ , then whereas A and B (a) are the least in their proportion, A (b) shall measure B as many times as B does C; but A (c) measures it self also; therefore A and B are not prime to one another, against the hypothesis.

## P R O P. XVII.

A, 8. B, 12. C, 18. D, 27. E ---

If there are numbers how many soever in continual proportion A, B, C, D, and the extremes of them A, D, be prime one to another, the first A shall not be to the second B, as the last D to any other E.

a 23. 7.  
b 21. 7.  
c 20. def. 7.  
d 11. ax. 7.

Suppose  $A : B :: D : E$ , then alternately  $A : D :: B : E$ , therefore seeing A and B are (a) the least in their proportion, A (b) shall measure B, (c) and B likewise C, and C the following number D; (d) and so A shall measure the said number D. Wherefore A and D are not prime to one another; contrary to the hypothesis.

## P R O P. XVIII.

A, 4. B, 6. C, 9.

Bq, 36.

Two numbers being given A, B, to consider if there may be a third number C found proportional to them.

a 9. ax. 7.  
b 20. 7.

If A measures Bq by any number C, (a) then  $AC = Bq$ , from whence (b) it is manifest that  $A : B :: B : C$ . Which was to be done.

But if A does not measure Bq, A, 6. B, 4. Bq, 16. there will not be any third proportional. For suppose  $A : B$

c 7. ax. 7.

$:: B : C$ , (a) then  $AC = Bq$ , (c) and consequently  $\frac{Bq}{A} = C$ , namely A measures Bq. Which is against the Hypothesis.

## P R O P. XIX.

A, 8. B, 12. C, 18. D, 27.  
BC, 216.

Three numbers being given A, B, C, to consider if a fourth proportional to them D may be found.

a 9. ax. 7.  
b ax. 19. 7.

If A measures BC by any number D, (a) then  $AD = BC$ ; (b) therefore it appears that  $A : B :: C : D$ , which was required.

But if A does not measure BC, then there can no fourth proportional be found; which may be shewn as in the last prop.

PROP. XX.

More prime numbers may be given than any multitude whatsoever of prime numbers A, B, C, propounded.

A, 2. B, 3. C, 5.  
D, 30. G ----

(a) Let D be the least which A, B, C, measure; If D + 1 be a prime, the case is plain: if composed, (b) then some prime number, suppose G, measures D + 1, which is none of the three A, B, C; For if it be, seeing it (c) measures the whole D + 1, (d) and the part taken away D, (e) it shall also measure the remaining unit. Which is absurd. Therefore the propounded number of prime numbers is increased by D + 1, or at least by G.

a 38. 7.

b 33. 7.

c suppos.

d constr.

e 12. ax. 7.

PROP. XXI.

<sup>5</sup>A.....<sup>5</sup>E.....<sup>3</sup>B...<sup>3</sup>F...<sup>2</sup>C...<sup>2</sup>G...D 20.

If even numbers, how many soever AB, BC, CD, are added together, the whole AD shall be even.

(a) Take  $EB = \frac{1}{2} AB$ , and  $FC = \frac{1}{2} BC$ , and  $GD = \frac{1}{2} CD$ ; (b) it is plain that  $EB + FC + GD = \frac{1}{2} AD$ , (c) therefore AD is an even number. Which was to be demonstrated.

a 6. def. 7.

b 12. 7.

c 6 def. 7.

PROP. XXII.

<sup>1</sup>A.....<sup>1</sup>F.B.....<sup>1</sup>G.C.....<sup>1</sup>H.D...<sup>1</sup>L.E 22.

If odd numbers, how many soever, AB, BC, CD, DE, are added together, and the multitude of them be even, the whole also AE shall be even.

Unity being taken from each odd number, there will (a) remain AF, BG, CH, DL, even numbers, (b) and thence the number compounded of them will be even, add to them the (c) even number made of the remaining units, and the (d) whole AE will thereby be even. Which was to be demonstrated.

a 7. def. 7.

b 21. 9.

c hyp.

d 21. 9.

## P R O P. XXIII.

$\overset{7}{A} \dots \dots \overset{5}{B} \dots \dots \overset{1}{C} \dots \overset{1}{E} \cdot \overset{3}{D} \text{ } 15.$

If odd numbers how  
many soever, AB, BC,  
CD, are added toge-  
ther, and the multitude

of them be odd, the whole AD shall be odd.

a 22. 9.  
b 21. 9.  
c 7. def. 7.

For CD, one of the odd numbers, being taken away, the aggregate of the others AC (a) is even. Whereto add CD — 1, (b) the whole AE is also even; wherfore the unit being restored the whole AD (c) will be odd. Which was to be demonstrated.

## P R O P. XXIV.

$\overset{4}{A} \dots \dots \overset{5}{B} \dots \dots \overset{1}{D} \cdot \overset{1}{C} \text{ } 10.$

If an even number AB be  
taken away from an even  
number AC, that which re-  
mains BC shall be even.

a 7. def. 7.  
b hyp.  
c 21. 9.

For if BD (BC—1) be odd, (a) BC (BD+1) will be even. Which was to be demonstrated. But if you say BD even, because AC (b) is even, (c) thence AD will be so; (a) and consequently AC (AD+1) will be odd, contrary the Hypothesis, therefore BC is even. Which was to be demonstrated.

## P R O P. XXV.

$\overset{6}{A} \dots \dots \overset{1}{D} \cdot \overset{3}{C} \dots \overset{1}{B} \text{ } 10.$

If from an even number AB,  
an odd number AC be taken  
away, the remaining number  
CB shall be odd

a 7. def. 7.  
b 24. 9.  
c 7. def. 7.

For AC—1 (AD) (a) is even (b) therefore DB is even; (c) and consequently CB (DB—1) is odd. Which was to be demonstrated.

## P R O P. XXVI.

$\overset{4}{A} \dots \overset{6}{C} \dots \dots \overset{1}{D} \cdot \overset{1}{B} \text{ } 11.$

If from an odd number AB  
be taken away an odd number  
CB, that which remaineth  
AC shall be even.

For



For  $AB - 1$  ( $AD$ ) and  $CB - 1$  ( $CD$ ) ( $a$ ) are even ;  $a$  7. def. 7.  
 (b) therefore  $AD - CD$  ( $AC$ ) is even. Which was to be  $b$  24. 9.  
 demonstrated.

PROP. XXVII.

If from an odd number  
 $AB$  be taken away an even  
 number  $CB$ , the residue  $AC$   
 shall be odd.

$$\begin{array}{r} 1 \quad 4 \quad 6 \\ A.D \dots C \dots B \text{ 11.} \\ 5 \end{array}$$

For  $AB - 1$  ( $DB$ ) ( $a$ ) is even, and  $CB$  is supposed to be  
 even : (b) therefore the residue  $CD$  is even : (c) therefore  
 $CD + 1$  ( $CA$ ) is odd, Which was to be demonstrated.

$a$  7. def. 7.  
 $b$  24. 9.  
 $c$  7. def. 7.

PROP. XXVIII.

If an odd number  $A$ , multiplying an even  
 number  $B$ , produces a number  $AB$ , the number  
 produced  $AB$  shall be even.

$$\begin{array}{r} A, 3 \\ B, 4 \\ \hline AB, 12. \end{array}$$

For  $AB$  ( $a$ ) is compounded of the odd number  $A$  taken  
 as many times as an unit is contained in  $B$ , an even  
 number. (b) Therefore  $AB$  is an even number. *W.W.D.*

$a$  hyp. and  
 15. def. 7.  
 $b$  21. 9.

*Schol.*

In like manner, if  $A$  be an even number,  $AB$  shall be  
 an even number also.

PROP. XXIX.

If an odd number  $A$  multiplying an odd num-  
 ber  $B$ , produces a number  $AB$ , the number produ-  
 ced  $AB$ , shall be odd.

$$\begin{array}{r} A, 3. \\ B, 5. \\ \hline AB, 15. \end{array}$$

For  $AB$  ( $a$ ) is compounded of the odd number  $B$  taken  
 as often as an unit is included in  $A$ , likewise an odd num-  
 ber. (b) Therefore  $AB$  is an odd number. Which was to  
 be demonstrated.

$a$  15. def. 7.  
 $b$  23. 9.

*Schol.*

1. An odd number  $A$  measuring an even  
 number  $B$ , measures the same by an even  
 number  $C$ .

$$\begin{array}{r} B, 12. \quad (C, 4 \\ A, 3. \end{array}$$

For if  $C$  be affirmed to be odd, then because ( $a$ )  $B =$   
 $AC$ , (b) therefore  $B$  shall be odd ; against the hyp.

$a$  9 ax. 7.  
 $b$  29. 9.

2. An

$\frac{B, 15.}{A, 3.}$  (C, 5.

2. An odd number A measuring an odd number B, measures the same by an odd number C.

a 28. 9.

For if C be said to be even, (a) then AC, or B will be even, contrary to the Hypothesis.

$\frac{B, 15.}{A, 3.}$  (C, 5.

3. Every number (A and C) that measures an odd number B, is it self an odd number.

a 28. 9.

For if either A or C be affirmed to be even, B (a) shall be an even number, against the Hypothesis.

P R O P. XXX.

$\frac{B, 24.}{A, 3.}$

(C, 8.

$\frac{D, 12.}{A, 3.}$

(E, 4.

If an odd number A measures an even number B, it shall also measure the half of it D.

a hyp.

b 1 schol.

29. 9.

c 9. ax. 7.

d 1. 2.

e hyp.

f 7. ax. 1.

g 7. ax. 7.

(a) Let  $\frac{B}{A}$  be = C; (b) then C is an even number.

Therefore let E be =  $\frac{1}{2}$  C, then B (c) = CA (d) = 2 EA

(e) = 2 D, (f) therefore EA = D; (g) and consequently  $\frac{B}{A}$

= E. Which was to be demonstrated.

P R O P. XXXI.

A, 5. B, 8. C, 16. D ---

If an odd number A be prime to any number B, it shall

also be prime to the double thereof C.

a 3. schol.

29. 9.

b 30. 9.

If it be possible, let some number D measure A and C, (a) then D measuring the odd number A shall be odd it self, (b) and so shall measure B, the half of the even number C, therefore A and B are not prime one to another. Which is against the Hypothesis.

Coroll.

It follows from hence, that an odd number which is prime to any number of a double progression, is also prime to all the numbers of that progression.

PROP. XXXII.

All numbers A, B, C, D, 1. A, 2. B, 4. C, 8. D, 16.  
 &c. in double progression  
 from two, are evenly even only.

It is evident that all these numbers A, B, C, D, (a) are even, and (b)  $\therefore$ , namely in a double proportion, (c) and every less measures the greater by some one of them. (d) Wherefore all are evenly even. But because A is a prime number, (e) no number besides these shall measure any of them. Therefore they are evenly even only. Which was to be demonstrated.

a 6. def. 7.  
 b 20. def. 7.  
 c 11. 9.  
 d 8. def. 7.  
 e 13. 9.

PROP. XXXIII.

If of a number A, the half of B be odd, A, 30. B, 15.  
 the same A is evenly odd only. D --- E ---

Since an odd number B (a) measures A by 2 an even number, (b) therefore B is evenly odd. If you affirm it to be evenly even, (c) then some even number D measures it by an even number E, whence  $2B$  (d)  $= A$  (d)  $= DE$ ; (e) therefore  $2 : E :: D : B$ , and therefore as 2 (f) measures the even number E, (g) so D an even number measures B an odd. Which is impossible.

a hyp.  
 b 9. def. 7.  
 c 8. def. 7.  
 d 9. ax. 7.  
 e 19. 7.  
 f 6. def. 7.  
 g 20. def. 7.

PROP. XXXIV.

If an even number A be neither doubled from two, nor have it's half odd, it is both evenly even and evenly odd. A, 24.

It is evident that A is evenly even, because the half of it is not odd. But because, if A be divided into two equal parts, and the half of it be again divided into two equal parts, and if this be always done, we shall at length fall upon some (a) odd number, (for we cannot fall upon the number two, because A is not supposed to be doubled upward from two) which shall measure A by an even number, for (b) otherwise A it self should be odd; against the Hyp. Therefore A is evenly odd. Which was to be demonstrated.

a 7. def. 7.  
 b 1. sch. 29.  
 9:

PROP.



## P R O P. XXXV.

$A \dots\dots\dots 8.$   
 $\quad \quad \quad 4 \quad \quad \quad 8$   
 $B \dots F \dots\dots\dots G \quad 12.$   
 $C \dots\dots\dots 18.$   
 $\quad \quad \quad 9 \quad \quad \quad 6 \quad \quad \quad 4 \quad \quad \quad 8$   
 $D \dots\dots\dots H \dots\dots\dots L \dots\dots\dots K \dots\dots\dots N \quad 27.$

If there are numbers in continual proportion how many soever A, BG, C, DN, and the number FG, equal to the first A, be taken from the second, and KN, also equal to the first, from the last; it shall be as the excess of the second BF is to the first A, so is the excess of the last DK to all the numbers that precede it, A, BG, C.

From DN take NL = BG, and NH = C. Because DN : C (HN) (a) : : HN : BG (LN) (a) : : LN (BG) : A (KN); (b) therefore by dividing every where, it will be DH : HN : : HL : LN : : LK : KN; (c) wherefore DK : C + BG + A : : LK (d) (BF) : KN (A.) Which was to be demonstrated.

## Coroll.

Hence (c) by compounding, DN + BG + C : A + BG + C : : BG : A.

## P R O P. XXXVI.

$1. A, 2. B, 4. C, 8. D, 16.$   
 $E, 31. G, 62. H, 124. L, 248. F, 496.$   
 $M, 31. N, 465.$   
 $P \dots\dots\dots Q \dots\dots\dots$

If from unity be taken how many numbers soever 1, A, B, C, D, in double proportion continually, until the whole added together E be a prime number; and if this whole E multiply'd into the last D, produce a number F, that which is produced F, shall be a perfect number.

Take as many numbers E, G, H, L, likewise in double proportion continually; then (a) by equality A : D : : E : L; (b) therefore AL = DE (c) = F, (d) whence L =  $\frac{F}{2}$ . Wherefore E, G, H, L, F, are :: in double proportion. Let G - E be = M, and F - E = N; (e) then M : E : : N : E + G + H + L. (f) But M = E, (g) there

a hyp.  
b 17. 5.  
c 12. 5.  
d 7. ax. 1.

e 18. 5.

a 14. 7.  
b 19. 7.  
c hyp  
d 7. ax. 7.  
e 35. 9.  
f 3. ax. 1.  
g 14. 5.

therefore  $N = E + G + H + L$ ; (b) therefore  $F = 1 + A + B + C + D + E + G + H + L = E + N$ . Moreover because D (k) measures DE (F) (l) therefore every one, A, B, C, (m) measuring D, and (m) also E, G, H, L, measures F. And further, no other number measures the said F. For if there does, let it be P, which let measures F by Q, (n) therefore  $PQ = F = DE$ ; (o) therefore  $E : Q :: P : D$ , therefore seeing A a prime number, measures D, (p) and so no other P measures the same, (q) consequently E does not measure Q. Wherefore since E is supposed a prime number, (r) it shall be prime to Q, (s) wherefore E and Q are the least in their proportion; (t) and so E measures P as many times as Q does D: (u) therefore Q is one of them A, B, C. Let it be B, seeing then by equality  $B : D :: E : H$ , (x) and so  $BH = DE = F = PQ$ , (x) and so also  $Q : B :: H : P$ , (y) therefore  $H = P$ ; therefore P is also one of them A, B, C, &c. *Against the Hypothesis*. Therefore no other beside the foresaid numbers measure F, and (z) consequently F is a perfect number. *Which was to be demonstrated.*

h 2. ax. 1.  
k 7. ax. 7.  
l 11. ax. 7.  
m 11. 9.  
  
n 9. ax. 7.  
o 19. 7.  
p 13. 9.  
q 20. def. 7.  
r 31. 7.  
s 23. 7.  
t 21. 7.  
u 13. 9.  
x 19. 7.  
y 14. 5.  
  
z 22. def. 7.

*The End of the ninth Book.*



The

# The TENTH BOOK OF EUCLIDE'S ELEMENTS.

## Definitions.

- I. **M**agnitudes are said to be commensurable which one and the same measure, measures them all.

Plate IV,  
Fig. 10.

The note of commensurability is  $\sqcup$ , as  $A \sqcup B$ ; that is, the line  $A$  of 8 foot is commensurable to the line  $B$  of 13 foot; because  $D$  a line of one foot measures both  $A$  and  $B$ . Also  $\sqrt{18} \sqcup \sqrt{50}$ , because  $\sqrt{2}$  measures both  $\sqrt{18}$  and  $\sqrt{50}$ . For  $\sqrt{\frac{1}{2}} = \sqrt{9} = 3$  and  $\sqrt{\frac{5}{2}} = \sqrt{25} = 5$ ; wherefore  $\sqrt{18} : \sqrt{50} :: 3 : 5$ .

II. Incommensurable magnitudes are such, of which no common measure can be found.

Incommensurability is denoted by this mark  $\sqcup$ ; as  $6 \sqcup \sqrt{25} (5)$ ; that is,  $\sqrt{6}$  is incommensurable to the number 5, or to a magnitude designed by that number; because there is no common measure of them, as shall appear hereafter.

III. Right-lines are commensurable in power, when the same space measures their squares.

The mark of this commensurability is  $\sqsupset$ ; as  $AB \sqsupset CD$ , i. e. the line  $AB$  of 6 foot is in power commensurable to the line  $CD$ , which is expressed by  $\sqrt{20}$ , because  $E$  the space of one foot square does as equal measure  $ABq$  (36) as the rectangle  $XY$  (20) to which the square of the line  $CD$  ( $\sqrt{20}$ ) is equal. The same note  $\sqsupset$  sometimes signifies commensurable in power only.

Fig. 11.



IV. Lines incommensurable in power are such, to whose squares no space can be found to be a common measure.

This incommensurability is denoted thus;  $5 \nmid \sqrt{8}$ . i. e. the numbers or lines 5, and  $\sqrt{8}$  are incommensurable in power, because their squares 25 and 8 are incommensurable.

V. From which it is manifest, that to any right-line given, right-lines infinite in number are both commensurable and incommensurable; some in length and power, others in power only. The right-line given is called a Rational-line.

The note of which is  $\rho$ .

VI. And lines commensurable to this line, whether in length and power, or in power only, are also called Rational  $\rho$ .

VII. But such as are incommensurable to it, are called Irrational,

And denoted thus  $\rho$ .

VIII. Also the square which is made of the said given right-line is called Rational,  $\rho$ .

IX. And likewise such figures as are commensurable to it, are Rational  $\rho$ .

X. But such as are incommensurable, Irrational  $\rho$ .

XI. And those right-lines also, which contain them in power, are Irrational  $\rho$ .

Schol. Plate IV. Fig. 12.

That the last seven definitions may be rendered more clear by an example, let there be a circle ADBP, whose semidiameter is CB, inscribe therein the sides of the ordinate figures, as of a Hexagon BP, of a triangle AP, of a square BD, of a Pentagon FD. Therefore, if according to the 5. def the semidiameter CB be the Rational line given, expressed by the number 2, to which the other lines BP, AP, BD, FD, are to be compared, then BP (a) = BC = 2, wherefore BP is  $\rho \nmid$  BC, according to the 6. def. Also AP (b) =  $\sqrt{12}$  (for ABq (16) - BPq (4) = 12) therefore AP is  $\rho \nmid$  BC, likewise according to the 6. def. and APq (12) is  $\rho$  by the 9. def. Moreover BD (b) =  $\sqrt{DCq + BCq} = 8$ ; whence BD is  $\rho \nmid$  BC; and BDq  $\rho$ . Lastly, FDq = 10 -  $\sqrt{20}$  (as shall appear by the axis to be delivered at the 10. 13.) shall be  $\rho$ , according to the 10. def. and consequently FD =  $\sqrt{10 - \sqrt{20}}$  is  $\rho$ , according to the 11. def.

a cor. 15. 4.  
b 47. 1.

A Postulate

*A Postulate.*

That any magnitude may be so often multiplied, till it exceed any magnitude whatsoever of the same kind.

*Axioms.*

1. A magnitude measuring how many magnitudes soever, does also measure that which is composed of them.
2. A magnitude measuring any magnitude whatsoever, does likewise measure every magnitude which that measures.
3. A magnitude measuring a whole magnitude and a part of it taken away, does also measure the residue.

## P R O P. I. Plate IV. Fig. 13.

Two unequal magnitudes AB, C, being given, if from the greater AB there be taken away more than half (AH) and from the residue (HB) be again taken more than half (HI) and this be done continually, there shall at length be left a certain magnitude IB, less than the lesser of the magnitudes first given C.

a post. 10.

(a) Take C so often, till its multiple does somewhat exceed AB, and let  $DF = FG = GE = C$ . Take from AB more than half AH, and from the remainder HB, more than half, viz. HI, and so continually, till the parts AH, HI, IB, be equal in multitude to the parts DF, FG, GE. Now it is plain, that FE which is not less than  $\frac{1}{2}$  DE, is greater than HB, which is less than  $\frac{1}{2}$  AB  $\supset$  DE. And in like manner GE, which is not less than  $\frac{1}{2}$  FE, is greater than IB  $\supset$   $\frac{1}{2}$  HB, therefore C, or GE  $\supset$  IB. Which was to be demonstrated.

The same may also be demonstrated, if from AB the half AH be taken away, and again from the residue HB the half HI, and so forward.

## P R O P. II. Fig. 14.

Two unequal magnitudes being given (AB, CD) if less AB be continually taken from the greater CD, by an interchangeable subtraction, and the residue do not measure the magnitude going before, then are the magnitudes given incommensurable.

If it be possible, let some magnitude E be the common measure. Then because AB taken from CD, as often as it can be, leaves a magnitude FD less than it self, and FD taken from AB leaves GB, and so forward (a) therefore at length some magnitude GB  $\supset$  E shall be left, therefore E (b) measuring AB, (c) and so CF, (b) and the whole CD, (d) shall also measure the residue FD, (c) consequently also AG; (d) wherefore it shall likewise measure the remainder GB, less than it self. Which is absurd.

a 1. 10.  
b hyp.  
c 2. ax. 10.  
d 3 ax. 10.

P R O P. III. Plate IV. Fig. 15.

Two commensurable magnitudes being given AB, CD, to find out their greatest common measure FB.

Take AB from CD, and the residue ED from AB, and FB from ED, till FB measure ED (which will come to pass at length, (a) because by the Hyp. AB  $\supset$  CD) FB shall be the magnitude required.

a 2. 10.

For FB (b) measures ED, (c) and so also AF; but it measures it self too, (d) therefore likewise AB, (c) and consequently CE, (d) and so the whole CD. Wherefore FB is the common measure of AB, CD. If you affirm G to be a common measure greater than that, then G measuring AB and CD, (e) measures also CE and (f) the remainder ED, (e) and so AF; and (f) consequently the remainder FB, the greater the less. Which is absurd.

b const.  
c 2. ax. 10.  
d 1. ax. 10.

e 2. ax. 10.  
f 3. ax. 10.

Coroll.

Hence, a magnitude that measures two magnitudes, does also measure their greatest common measure.

P R O P. IV. Fig. 16.

Three commensurable magnitudes being given, A, B, C, to find out their greatest common measure.

(a) Find out D the greatest common measure of any two A, B; (a) also E the greatest common measure of D and C, therefore E is the magnitude sought.

a 3. 10.  
2 ax. 10.

(a) For it is clear, E measuring D and C, (b) measures the three, A, B, C. Conceive another magnitude F greater than that to measure them; (c) then F measures D, (c) and consequently E the greatest common measure of D, and C, the greater the less. Which is absurd.

b const. &  
c cor. 3. 10.

K

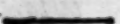
Coroll.



## Coroll.

Hence also it appears, that if a magnitude measures three magnitudes, it shall likewise measure their greatest common measure.

## P R O P. V.

A  D, 4. Commensurable magni-  
C  F, 1. tudes A, B, have such pro-  
B  E, 3. portion one to another as  
number bath to number.

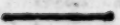
a 3. 10.

b 20. def. 7.

c 22. 5.

(a) C being found the greatest common measure of A, B; as often as C is contained in A and B, so often is 1 contained in the numbers D and E; (b) therefore  $C : A :: 1 : D$ ; wherefore inversely  $A : C :: D : 1$ , (b) but likewise  $C : B :: 1 : E$ ; (c) therefore by equality  $A : B :: D : E :: N : N$ . Note, The letter N only signifies number in general, and refers not to any particular space or magnitude as the other letters do, and is to be read, as A to B, so D to E, and so number to number. Which was to be demonstrated.

## P R O P. VI.

E  F, 1. If two magnitudes A,  
A  C, 4. B, have such proportion  
B  D, 3. one to another, as the num-  
ber C bath to the number

D, those magnitudes A, B, shall be commensurable.

a sch. 10. 6.

b constr.

c hyp.

d 22. 5.

e 5. ax. 7.

f 20. def. 7.

g const.

h 1. def. 10.

What part 1 is of the number C, (a) that let E be of A. Therefore because  $E : A (b) :: 1 : C$ , and  $A : B (c) :: C : D$ , (d) therefore by equality shall  $E : B :: 1 : D$ . Wherefore seeing 1 (e) measures the number D, (f) likewise E measures B; but it (g) also measures A, (h) therefore  $A \sqsubset B$ , Which was to be demonstrated.

## P R O P. VII. Plate IV. Fig. 17.

Incommensurable magnitudes A, B, have not that proportion one to another, which number bath to number.

a 6. 10.

If you affirm  $A : B :: N : N$ , (a) then  $A \sqsubset B$ , against the Hypothesis.

P R O P.

P R O P. VIII. Plate IV. Fig. 17.

If two magnitudes A, B, have not that proportion one to another, which number hath to number, those magnitudes are incommensurable.

Conceive  $A \nmid B$ , (a) then  $A : B :: N : N$ , contrary to the Hypothesis.

a 5. 10.

P R O P. IX.

The squares described upon right lines commensurable in length, have that proportion one to another; that a square number hath to a square number: And squares, which have that proportion one to another; that a square number hath to a square number; shall also have their sides commensurable in length. But such squares as are made upon right-lines incommensurable in length, have not that proportion one to another; which a square number hath to a square number: And squares which have not such proportion one to another; as a square number hath to a square number; have not their sides commensurable in length.

A ———  
B ———  
E, 4.  
F, 3.

1. Hyp.  $A \nmid B$ . I say that  $Aq : Bq :: Q : Q$ .  
For (a) let  $A : B ::$  number E : number F; therefore  
 $\frac{Aq}{Bq} (b) \frac{A}{B}$  twice (c)  $= \frac{E}{F}$  twice, (d)  $= \frac{Eq}{Fq}$ , (e) therefore  $Aq :$   
 $Bq :: Eq : Fq :: Q : Q$ . Which was to be demonstrated.

a 5. 10.  
b 20. 6.  
a sch. 23. 5.  
d 11. 8.  
e 11. 5.

2. Hyp.  $Aq : Bq :: Eq : Fq :: Q : Q$ . I say  $A \nmid B$ . For  
 $\frac{A}{B}$  twice (f)  $\frac{Aq}{Bq} (g) = \frac{Eq}{Fq} (h) = \frac{E}{F}$  twice, (i) therefore  $A :$   
 $B :: E : F :: N : N$ , (k) wherefore  $A \nmid B$ . Which was to be demonstrated.

f 20. 6.  
g hyp.  
h 11. 8.  
i sch. 23. 5.  
k 6. 10.

3. Hyp.  $A \nmid B$ . I deny that  $Aq : Bq :: Q : Q$ . For suppose  $Aq : Bq :: Q : Q$ , then  $A \nmid B$ , as is shewn before, against the Hypothesis.

4. Hyp. Not  $Aq : Bq :: Q : Q$ , I say that  $A \nmid B$ . For conceive  $A \nmid B$ , then  $Aq : Bq :: Q : Q$ , as above, against the Hypothesis.

Coroll.

Lines  $\nmid$  are also  $\nmid$ , but not on the contrary. And lines  $\nmid$  are not therefore  $\nmid$ , but Lines  $\nmid$  are also  $\nmid$ .

K 2

P R O P.

## PROP. X. Plate IV. Fig. 18.

If four magnitudes are proportional ( $C : A :: B : D$ ) and the first  $C$  be commensurable to the second  $A$ , the third  $B$  shall be commensurable to the fourth  $D$ . And if the first  $C$  be incommensurable to the second  $A$ , also the third  $B$  shall be incommensurable to the fourth  $D$ .

a 5. 10.  
b 6. 10.  
c 7. 10.  
d 8. 10.

If  $C \nmid A$ , (a) then  $C : A :: N : N$ , (b)  $:: B : D$ , (b) therefore  $B \nmid D$ . But if  $C \mid A$ , (c) then shall not  $C : A :: N : N :: B : D$ , (e) wherefore  $B \nmid D$ . Which was to be demonstrated.

## Lemma 1.

To find out two plane numbers, not having the proportion which a square number hath to a square.

Any two plane numbers not like, will satisfy this Lemma, as those numbers which have super-particular, superbipartient, or double proportion; or any two prime numbers, See Schol. 27. 8.

## Lemma 2. Fig. 19.

To find out a line  $HR$ , to which a right-line given  $KM$  hath the proportion of two numbers given  $B, C$ .

a sch 10. 6.  
b 3. 1.

(a) Divide  $KM$  into as many equal parts as there are units in the number  $B$ , and let as many of these, as there are units in the number  $C$ , (b) make the right-line  $HR$ , it is manifest that  $KM : HR :: B : C$ .

## Lemma 3.

To find out a line  $D$ , to the square of which the square of a right-line given  $KM$  hath the proportion of two numbers given  $B, C$ .

a 2. lem. 10.  
10.  
b 13. 6.  
c 20. 6.  
d constr.

Make  $B : C$  (a)  $:: KM : HR$ , and between  $KM$  and  $HR$ , (b) find a mean proportional  $D$ . Therefore  $KM : Dq$  (c)  $:: KM : HR$  (d)  $:: B : C$ .

## PROP. XI. Fig. 20.

a 1. lem. 10. To find two right-lines incommensurable to a right-line given  $A$ , one  $D$  in length only, the other  $E$  in power also.  
10.  
b 3. lem. 10. 1. Take the numbers  $B, C$ , (a) so that it be not  $B : C :: Q : Q$ , (b) and let  $B : C :: Aq : Dq$ , (c) it is plain that  $A \nmid D$ . But  $Aq$  (d)  $\nmid Dq$ . Which was to be done.  
c 9. 10.  
d 6. 10. 2. (d)



2. (d) Make  $A : E :: E : D$ . I say  $Aq \sqsubset Eq$ . For  $d$  13. 6.  
 $A : D(e) :: Aq : Eq$ , therefore since  $A \sqsubset D$ , as before;  $e$  20. 6.  
 (f) therefore  $Aq \sqsubset Eq$ . Which was to be done.  $f$  10. 10.

PROP. XII. Plate IV, Fig. 20.

Magnitudes (A,B) commensurable to the same magnitude C, are also commensurable one to the other.

Because  $A \sqsubset C$ , and  $C \sqsubset B$ , (a) let  $A : C :: M : N$   $a$  5. 10.  
 $D, 8. E, 8. :: D : E$ , and  $C : B :: M : N ::$   
 $F : G$ , (b) take three numbers H,  $b$  4. 8.  
 $F, 2. G, 3. I, K$ , the least in the proportion  
 $H, 5. I, 4. K, 6.$  of D to E, and F to G. Now because  
 $A : C(c) :: D : E(c) :: H : I$ , and  $C : B(c) :: F : G$   $c$  constr.  
 $:: I : K$ , (d) therefore by equality,  $A : B :: H : K ::$   $d$  22. 5.  
 $M : N$ , (e) therefore  $A \sqsubset B$ . Which was to be de-  $e$  6. 10.  
 monstrated.

Schol.

Hence, every right-line commensurable to a rational line is also it self rational. And all rational right-lines are commensurable to one another, at least in power.  $12. 10. \text{ } \text{\textcircled{C}}$   
 Also every space commensurable to a rational space is rational too: And all rational spaces are commensurable one to another. But magnitudes whereof one is rational, the other irrational, are incommensurable amongst themselves.  $def. 6.$   
 $def. 9.$   
 $def. 7. and$   
 $10.$

PROP. XIII. Fig. 21.

If there are two magnitudes A, B, and one of them A, commensurable to a third C, but the other B incommensurable, those magnitudes AB, are incommensurable.

Conceive  $B \sqsubset A$ , then since  $C(a) \sqsubset A$ , (b) therefore  $C \sqsubset B$ , against the Hypothesis.  $a$  hyp.  
 $b$  12. 10.

PROP. XIV. Fig. 20.

If there are two magnitudes commensurable A, B, and one of them A incommensurable to any other magnitude C, the other also B shall be incommensurable to the same C.

Imagine  $B \sqsubset C$ , then for that  $A(a) \sqsubset B$ , (b) therefore  $A \sqsubset C$ , against the Hyp.  $a$  hyp.  
 $b$  12. 10.

K 3

PROP.

## P R O P. XV. Plate IV. Fig. 22.

If four right-lines are proportional ( $A : B :: C : D$ ) and the first  $A$  be in power more than the second  $B$  by the square of a right-line commensurable to it self in length, then also the third  $C$ , shall be more in power than the fourth  $D$  by the square of a right-line commensurable to it self in length. But if the first  $A$  be more in power than the second  $B$ , by the square of a right-line incommensurable to it self in length, then shall the third  $C$  be more in power than the fourth  $D$  by the square of a right-line incommensurable to it self in length.

a hyp. For because  $A : B (a) :: C : D$ , (b) therefore  $Aq : Bq :: Cq : Dq$ , (c) therefore by division  $Aq - Bq : Bq :: Cq - Dq : Dq$ ; (d) wherefore  $\sqrt{Aq - Bq} : B :: \sqrt{Cq - Dq} : D$ , (e) and so inversely  $B : \sqrt{Aq - Bq} :: D : \sqrt{Cq - Dq}$ , (f) therefore by equality  $A : \sqrt{Aq - Bq} :: C : \sqrt{Cq - Dq}$ , consequently if  $A \sqsupset$ , or  $\sqsubset \sqrt{Aq - Bq}$ , (g) then likewise  $C \sqsupset$ , or  $\sqsubset \sqrt{Cq - Dq}$ . Which was to be demonstrated.

a hyp.  
b 22. 6.  
c 17. 5.  
d 22. 6.  
e cor. 4. 5.  
f 22. 5.  
g 10. 10.

## P R O P. XVI. Fig. 23.

If two commensurable magnitudes  $AB$ ,  $BC$ , are composed, the whole magnitude  $AC$  shall be commensurable to each of the parts  $AB$ ,  $BC$ . And if the whole magnitude  $AC$  be commensurable to either of the parts  $AB$ , or  $BC$ , these two magnitudes given at first  $AB$ ,  $BC$ , shall be commensurable.

1. Hyp. (a) Let  $D$  be the common measure of  $AB$ ,  $BC$ ; (b) therefore  $D$  measures  $AC$ , and therefore  $AC \sqsupset$   $AB$ , and  $BC$ . Which was to be demonstrated.  
2. Hyp. (a) Let  $D$  be the common measure of  $AC$ ,  $AB$ , (d) therefore  $D$  measures  $AC - AB$  ( $BC$ ) and consequently  $AB \sqsupset$   $BC$ . Which was to be demonstrated.

a 3. 10.  
b 1. ax. 10.  
c 1. def. 10.  
d 3. ax. 10.

Coroll.

Hence it follows, if a whole magnitude composed of two, be commensurable to any one of them, the same shall be commensurable to the other also.

P R O P.

PROP. XVII. Plate IV. Fig. 23.

If two incommensurable magnitudes AB, BC, are composed, the whole magnitude also AC shall be incommensurable to either of the two parts AB, BC. And if whole magnitude AC be incommensurable to one of them AB, the magnitudes first given AB, BC, shall be incommensurable.

1. Hyp. If it can be, let D be the common measure of AC, AB, (a) therefore D measures  $\overline{AC} - \overline{AB}$  (BC) (b) and therefore also  $\overline{AB} \sqsupset \overline{BC}$ , against the hypothesis.

2. Hyp. Conceive  $\overline{AB} \sqsupset \overline{BC}$ , (c) therefore  $\overline{AC} \sqsupset \overline{AB}$ , against the hypothesis.

c 3. ax. 10.  
b 1. def. 10.  
c 16. 10.

Coroll.

Hence also, if one magnitude, composed of two, be incommensurable to any one of them, the same also shall be incommensurable to the other.

PROP. XVIII. Fig. 24.

If there are two unequal right-lines AB, GK, and upon the greater AB a parallelogram ADB equal to the fourth part of a square made of the less line GK, and deficient in figure by a square, be applied, and divides the said AB into parts commensurable in length AD, DB; then shall the greater line AB be more in power than the less GK by the square of a right-line FD commensurable in length to the greater. And if the greater AB be in power more than the less GK, by the square of the right-line FD commensurable in length to it self, and a parallelogram ADB equal to the fourth part of the square made of the less line GK, and deficient in figure by a square, be applied to the greater AB, then shall it divide the same into parts AD, DB, commensurable in length.

(a) Divide GK equally in H, and (b) make the rectangle  $\overline{ADB} = \overline{GHq}$ . Cut off  $\overline{AF} = \overline{DB}$ , then is  $\overline{ABq}$  (c)  $= 4 \overline{ADB}$  (d)  $(4 \overline{GHq}$  or  $\overline{GKq}) + \overline{FDq}$ . Now in the first place, if  $\overline{AD} \sqsupset \overline{DB}$ , then shall  $\overline{AB}$  (e)  $\sqsupset \overline{BD}$  (e)  $\sqsupset 2 \overline{DB}$  (f)  $(\overline{AF} + \overline{DB}$ , or  $\overline{AB} - \overline{FD})$  (g) therefore  $\overline{AB} \sqsupset \overline{FD}$ . Which was to be dem. But secondly, if  $\overline{AB} \sqsupset \overline{FD}$ , (h) then shall  $\overline{AB} \sqsupset \overline{AB} - \overline{FD}$  (2 DB) (k) therefore  $\overline{AB} \sqsupset \overline{DB}$ , (l) wherefore  $\overline{AD} \sqsupset \overline{DB}$ . Which was to be demonstrated.

a 10. 1.  
b 28. 6.  
c 8. 2.  
d constr. E  
4. 2.  
e 16 10.  
f constr.  
g cor. 16. 10.  
h cor. 16. 10.  
k 12. 10.  
l 16. 10.

K 4

PROP.



## P R O P. XIX. Plate IV. Fig. 24.

If there are two right-lines unequal AB, GK, and to the greater AB, a parallelogram ADB equal to the fourth part of a square made upon the less GK, and deficient in figure by a square be applied, and divides the said AB, into parts AD, DB, incommensurable in length; the greater line AB shall be in power more than the less GK by the square of the right-line FD incommensurable to the greater in length. And if the greater line AB be more in power than the less GK by the square of a right-line FD incommensurable to it in length, and if also upon the greater AB be applied a parallelogram ADB equal to the fourth part of the square of the less GK, and deficient in figure by a square, then shall it divide the said greater line AB, into parts incommensurable in length AD, DB.

a 17. 10.  
b 13. 10.  
c cor. 17.  
10.  
d 13. 10.  
e 17. 10.

Suppose all the same that was done and said in the prec. prop. Therefore first, If AD  $\square$  DB, (a) then shall AB  $\square$  DB. (b) Wherefore AB  $\square$  2 DB (AB - FD) (c) therefore AB  $\square$  FD. Which was to be demonstrated.

Secondly, If AB  $\square$  FD, then AB  $\square$  AB - FD (2 DB) (d) wherefore AB  $\square$  DB, (e) and consequently AD  $\square$  DB. Which was to be demonstrated.

## P R O P. XX. Fig. 25,

A rectangle BD comprehended under right-lines BC, CD, rational and commensurable in length according to one of the foresaid ways, is rational,

a 46. 1.  
b 1. 6.  
c hyp.  
d 10. 10.  
e hyp. &  
9. def. 10.  
f 12. 10.

Let A be given  $\rho$ , and (a) the square BE described upon BC. Because DC : CE (BC) (b) : : BD : BE, and DC (c)  $\square$  BC, (d) therefore shall the rectangle BD be  $\square$  square BE, wherefore seeing the square BE (e)  $\square$  Aq, shall also (f) BD be  $\square$  Aq, and so the rectangle  $\rho$  BD. Which was to be demonstrated.

Note, There are three kinds of rational-lines commensurable one to another. For either of two rational-lines commensurable in length one to the other, one is equal to the rational-line propounded, or either of them is equal to it, notwithstanding both of them are commensurable to it in length; or lastly both of them are commensurable to the rational-line given only in power. And these are the ways which the present Theorem speaks of.

In numbers, let there be BC,  $\sqrt{8}$  ( $2\sqrt{2}$ ) and CD  $3\sqrt{2}$  then shall the rectangle BD  $= \sqrt{144} = 12$

PROP. XXI. Plate IV. Fig. 26.

If a rational rectangle DB be applied to a rational DC, it makes the breadth thereof CB rational, and commensurable in length to that line DC whereto DB is applied.

Let G be propounded  $\rho$ , and the square DA described on DC, because BD : DA :: (a) BC : CA; and BD, (b) are  $\rho$  & (c) and so  $\square$  (d) therefore BC  $\square$  CA CD (CA) is  $\rho$ . (e) therefore BC is  $\rho$ . Which was to be demonstrated.

In numbers, let there be the rectangle DB, 12, and  $\sqrt{8}$ . then shall CB,  $\sqrt{18}$ . but  $\sqrt{18} = 3\sqrt{2}$ . and  $8 = 2 \times \sqrt{2}$ .

a 1. 6.

b hyp.

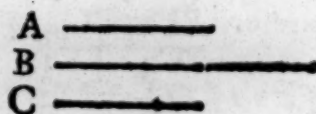
c sch. 12. 10.

d 10. 10.

e sch. 12. 10.

Lemma.

To find out two rational right-lines commensurable only in power.



Let A be propounded  $\rho$ . (a) make B  $\square$  A, (a) and C  $\square$  B, (b) it is clear that B and C are the lines required.

a 11. 10.

b sch. 12.

10.

PROP. XXII. Fig. 26.

A rectangle DB comprehended under the rational right-lines DC, CB commensurable in power only, is irrational: and the right-line H, which containeth that rectangle in power is irrational, and called a Medial-line.

Let G be the propounded  $\rho$ , and the square DA described on DC, and let Hq = DB. Because AC : CB (a) DA : DB, (b) and AC  $\square$  CB, (c) shall be DA  $\square$  DB (Hq.) (d) but Gq  $\square$  DA; (e) therefore Hq  $\square$  Gq wherefore H is  $\rho$ . Which was to be dem. and let it be called a Medial-line, because AC : H :: H : CB.

In numbers, let there be DC, 3. and CB,  $\sqrt{6}$ . then shall the rectangle be DB (Hq)  $\sqrt{54}$ . wherefore H is  $\sqrt{54}$ .

The note of a medial-line is  $\mu$ , of a medial-rectangle of both together  $\mu a$ .

a 1. 6.

b hyp.

c 10. 10.

d hyp. and

9 def. 10.

e 13. 10.

f 11. 10.

Schol.

Every rectangle that can be contained under two rational right-lines commensurable only in power, is medial, although it be contained under two right-lines irrational: and every medial-rectangle may be contained under two rational right-lines, commensurable only in power; as for example, the  $\sqrt{24}$  is  $\mu\nu$ , because it is contained under  $\sqrt{3}$ , and  $\sqrt{8}$ , which are  $\rho$ ,  $\square$  although it may be contained under  $\nu\sqrt{6}$ , and  $\nu\sqrt{96}$  irrationals for  $\sqrt{24} = \nu\sqrt{576} = \nu\sqrt{6 \times 96}$ .

## P R O P. XXIII. Plate IV. Fig. 27.

If the rectangle BD made of a medial-line A, be applied to a rational-line BC, it makes the breadth CD rational and incommensurable in length to the line BC, whereunto the rectangle BD is applied.

a sch. 22.

10.

b 1. ax. 1,

c 14. 6.

d 22. 6.

e hyp.

f sch. 12.

10.

g 10. 10.

h sch. 12.

10.

k 1. 6.

l 10. 10.

m sch. 12.

10.

n 13. 10.

o 1. 6.

p 10. 10.

a 11. 6.

b hyp.

c 23. 10.

d 1. 6.

e hyp.

f 10. 10.

g 12. and

13. 10.

h 22. 10.

Because A  $\mu$ , (a) therefore shall Aq be equal to some rectangle (EG) contained under EF and FG  $\rho$   $\square$ . (b) therefore BD = EG, (c) whence BC : EF :: FG : CD (d) therefore BCq : EFq :: FGq : CDq. But BCq and EFq (e) are  $\rho$  (f) and so  $\square$ . (g) Therefore FGq  $\square$  CDq. Wherefore since FG is  $\rho$ , (b) therefore CD shall be  $\mu$ . Moreover, because EF : FG (k) :: EFq : EG (BD) (l) since EF  $\square$  FG, (c) shall EFq be  $\square$  BD. But EF (m)  $\square$  CDq. (n) therefore the rectangle BD  $\square$  CDq. Whence since CDq : BD (o) :: CD : BC. (p) shall CD be  $\square$  BC. therefore, &c.

## P R O P. XXIV, Fig. 28.

A right-line B commensurable to a medial-line A, is a medial-line.

Upon CD  $\rho$  (a) make the rectangle CE = Aq; (a) and the rectangle CF = Bq. Because Aq (CE) is  $\mu\nu$ , (b) and CD  $\rho$ , (c) therefore shall the latitude DE be  $\rho$   $\square$  CD. But because CE : CF (d) :: ED : DF and CE (e)  $\square$  CD (f) therefore ED  $\square$  DF. (g) therefore DF is  $\rho$   $\square$  CD. whence the rectangle CF (Bq) is  $\mu\nu$ , and so B is  $\mu$ . Which was to be demonstrated.

Obs. that the note  $\square$  for the most part signifies commensurable in power only, as in this and the precedent demonstrations, &c.



Coroll.

whereby it is manifest that a space commensurable to a medial-space, is also medial.

Lemma. Plate IV. Fig. 21.

find out two medial right-lines A, B, commensurable in length and also two A C commensurable only in power.

(a) Let A be any  $\mu$ , (b) take B  $\perp$  A, and (c) C  $\perp$  B and 'tis evident the thing is done.

PROP. XXV. Fig. 29.

rectangle DB contained under DC, CB medial right-lines commensurable in length is medial.

Upon DC describe the square DA. Because AC : CB (a) :: DA : DB, and DC  $\perp$  CB ; (b) shall DC  $\perp$  DB. (c) therefore DB is  $\mu\nu$ . Which was to be demonstrated.

PROP. XXVI. Fig. 30.

rectangle AC comprehended under medial right-lines AB, commensurable only in power is either rational or medial.

Upon the lines AB, BC, (a) describe the squares AD, CE, and upon FG (b) make the rectangles FH = AC (b) and LM = CE.

The squares AD, CE, that is, the rectangles FH, LM, are  $\mu\alpha$  and  $\perp$ . therefore GH, KM, having the same proportion (d) are  $\rho$ , (e) and  $\perp$  (f) therefore GH  $\times$  KM are  $\perp$ . But because AD, AC, CE, that is FH, IK, LM, are  $\perp$ ; (b) and so GH, HK, KM also  $\perp$ ; (k) thence  $\perp$  = GH  $\times$  KM. (l) therefore HK is  $\rho$ , or  $\perp$ , or  $\mu$  (IH (GF;) if  $\perp$ , (m) then the rectangle IK or AC is  $\mu\nu$ , but if  $\perp$  (n) then AC is  $\mu\nu$ . Which was to be dem.

Lemma. Fig. 31.

If A and E are  $\perp$  only, Then first, shall Aq, Eq, Aq - Eq, Aq - Eq (a)  $\perp$ . And secondly Aq, Eq, Aq + Aq - Eq  $\perp$  AE and 2 AE. For A : E (b) :: Aq : (b) :: AE : Eq. therefore seeing A (c)  $\perp$  E, (d) Aq  $\perp$  AE, (e) and 2 AE. Also Eq (d)  $\perp$  AE, (e) 2 AE. wherefore because Aq + Eq  $\perp$  Aq and Eq; Aq - Eq  $\perp$  Aq and Eq; (f) therefore shall Aq + (f) and Aq - Eq be  $\perp$  AE, and 2 AE. Hence

a lem. 21.  
10 and 13.  
6.  
b 2. lem.  
10. 10.  
c 3. lem 10.  
10.  
d const. and  
24. 10.  
a 1. 6.  
b 10. 10.  
c 24. 10.

a 46. 1.  
b cor. 16. 6.  
c hyp. 8  
24. 10.  
d 23. 10.  
e 10. 10.  
f 20. 10.  
g sch. 22. 6.  
h 1. 6.  
k 17. 6.  
l 12. 10.  
m 20. 10.  
n 22. 10.

a hyp. and  
16. 10.  
b 1. 2.  
c hyp.  
d 10. 10.  
e 14. 10.  
f 14. 10.

g 14. 10.  
and 7. 10.  
h cor. 7. 2.

Hence also thirdly,  $Aq, Eq, Aq + Eq, Aq - Eq$   
 $AE (g) \sqsubset Aq + Eq + 2 AE$ ; and  $Aq + Eq -$   
 $AE. (g) \text{ and } Aq + Eq + 2 AE \sqsubset Aq + Eq -$   
 $AE. (b) (Q. A - E.)$

P R O P. XXVII. Plate IV. Fig. 32.

*A medial rectangle AB exceeds not a medial rectangle AC by a rational rectangle DB.*

a cor. 16. 6.

b hyp.

c 3. 10.

d 3. ax. 1.

e 1. 10.

f 13. 10.

g lem. 26.

10.

h sch. 12.

10.

Upon  $EF \rho$ , (a) make  $EG = AB$ , (a) and  $EH = AC$ .  
 The rectangles  $AB, AC$ , i. e.  $EG, EH$ , (b) are  $\mu$ .  
 therefore  $FG$  and  $FH$  are  $\rho \sqsupset EF$ . Whence, if  $K$   
 (d) i. e.  $DB$  be  $\rho$ , (e) then shall  $HG$  be  $\sqsubset HK$ ;  
 wherefore  $HG \sqsupset FH$ . (g) and consequently  $FG \sqsupset$   
 $FHq$ . But  $FH$  is  $\rho$ . (b) therefore is  $FG \rho$ . but  $FG$   
 $\rho$  before. Which is contradictory.

Schol. Fig. 29, 33.

1. *A rational rectangle AE exceeds a rational rectangle AD by a rational rectangle CE.*

a hyp.

b cor. 16.

10.

c sch. 12. 10.

a sch. 12. 10.

b 16. 10.

c sch. 12. 10.

For  $AE$  (a)  $\sqsubset AD$ , (b) therefore  $AE \sqsubset CE$   
 wherefore  $CE$  is  $\rho$ . Which was to be demonstrated.

2. *A rational rectangle AD joined with a rational rectangle CE makes a rational rectangle AF.*

For  $AD$  (a)  $\sqsubset CF$ , (b) wherefore  $AF \sqsubset AD$   
 $CF$ ; (c) and so  $AF$  is  $\rho$ . Which was to be demonstrated.

P R O P. XXVIII. Fig. 22.

*To find out medial-lines (C and D) which contain a rational rectangle CD.*

a lem. 21. 10.

b 13. 6.

c 12. 6.

d 22. 10.

e constr.

f 10. 10.

g 24. 10.

h 17. 6.

h sch. 12.

10.

(a) Take  $A$  and  $B \rho \sqsupset$ , (b) make  $A : C :: C : B$ .  
 and  $A : B :: C : D$  I say the thing required is done.  
 For  $AB (Cq) (d)$  is  $\mu$ , (d) whence  $C$  is  $\mu$ . But because  
 $A : B :: C : D (f)$  therefore  $C \sqsupset D$ ; (g) and consequently  $D$  is  $\mu$ .  
 Moreover by permutation  $A : C :: B : D$ . i. e.  $C : B :: B : D$ ;  
 (b) therefore  $Bq = CD$ . But  $Bq$  is  $\rho$ ; (b) therefore  $CD$  is  $\rho$ .  
 Which was to be done.

In numbers, let  $A$  be  $\sqrt{2}$ ; and  $B \sqrt{6}$ . therefore  
 is  $\sqrt{12}$ . make  $\sqrt{2} : \sqrt{6} :: \sqrt{12} : D$ . or  $\sqrt{4} : \sqrt{36} :: \sqrt{12} : D$ ;  
 then shall  $D$  be  $\sqrt{108}$ ; but  $\sqrt{12} : \sqrt{108} = \sqrt{1296} = \sqrt{36} = 6$ .  
 therefore  $CD$  is likewise  $C : D :: 1 : \sqrt{3}$ ; wherefore  $C \sqsupset D$ .

P R O

PROP. XXIX. Plate IV. Fig. 34.

find out medial right-lines commensurable in power only,  
E, containing a medial rectangle DE.

Take A, B, C  $\rho$   $\square$ , make A : D (b) : : D : B. (c)

B : C : : D : E I say the thing desired is performed.

AB (d) = Dq; and AB (e) is  $\mu r$ , therefore D is

and B (f)  $\square$  C, (g) whence D  $\square$  E. (b) E is  $\mu$ .

over B : C (f) : : D : E, and by permutation B : D

: E. i. e. D : A : : C : E; (l) therefore DE = AC.

AC (m) is  $\mu r$ ; therefore DE is  $\mu v$ . Which was to

be proved.

numbers, let A be 20, and B,  $\sqrt{200}$ , and C,  $\sqrt{80}$ .

Therefore D is  $\sqrt[4]{80000}$ ; and E  $\sqrt{12800}$ . There-

fore DE =  $\sqrt[4]{1024000000} = \sqrt{32000}$ . and D : E ::

2 : 1. wherefore D  $\square$  E.

Schol.

find out two plane numbers, like

like,

take any four numbers pro-

portional A : B : : C : D it is ma-

trix that AB and CD are like

numbers. And you may

put as many unlike plane

numbers, as you please, by the

Schol, 27.8.

|         |         |
|---------|---------|
| A, 6.   | C 12    |
| B, 4.   | D, 8.   |
| AB, 24. | CD, 96. |

|         |         |
|---------|---------|
| A, 6.   | C, 5.   |
| B, 4.   | D, 8.   |
| AB, 24. | CD, 40. |

Lemma. Fig. 35.

find out two square numbers (DEq and CDq) so that

number composed of them (CEq) be square also.

Take AD, DB like plane numbers (of which let both

be even, or both odd) viz. AD, 24. and DB 6. The

square of these (AB) is 30; the difference (FD) 18. Half

of these (CD) is 9. (a) Now the like plane numbers AD,

have one mean number proportional, namely DE;

and it is evident that every of those numbers CE,

DE, are rational, and by consequence CEq (b)

+ DEq) is the square number required.

whereby it will be easy to find out two square num-

bers the excess of which is a square or not a square

namely by the same construction (c) shall CEq -

DEq.

But

a 21. 10.  
b 13. 6.  
c 12. 6.  
d 17. 6.  
e 22. 10.  
f constr.  
g 10. 10.  
h 24. 10.  
k constr.  
and cor 4. 5.  
l 16. 6.  
m 22. 6.

a 18. 3.

b 47. 1.

c 3. ax. 1.



But if AD, DB be plane numbers unlike, the proportional line (DE) shall not be a rational number and so neither shall the excess (DEq) of the square numbers, CEq, CDq, be a square number.

Lemma 2.

2. To find out two such square numbers B, C, as number compounded of them D is not square, Also divide a square number A into two numbers B, C squares.

A, 3. B, 9. C, 36. D, 45.

1. Take any square number B, and let C be = and  $D = B + C$ . I say the thing is done.

For B is Q, by the constr. likewise because B : C :: 4 : 36 :: Q : Q; (a) therefore C also shall be a square number. But because  $B + C = D$  : C :: 5 : 4 :: Q : Q. (b) therefore shall not D be a square number. Which was to be done.

a 24. 8.  
b cor. 24. 8.

A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

2. Let A be some square number. Take D, E plane numbers unlike, and let D be =  $E + F$ .  $D : E :: A : B$ . and  $D : F :: A : C$ . I say the required is done.

For because  $D : E + F :: A : B + C$ , and  $D = F$ , (a) therefore shall  $A = B + C$ . Now suppose B be square, (b) then A and B, (c) and consequently D, E are like plane numbers. Which is contrary to Hyp.

a 14. 5.  
b 21. def. 7.  
c 26. 8.

The same absurdity will follow if C be supposed square number, Therefore, &c.

P R O P. XXX. Fig. 36.

To find out two rational right-lines AB, AF, commensurable only in power, so that the greater AB be in power, more than the less AF, by the square right-line BF, commensurable to it self in length.

a 1. lem. 29.  
10.

b 3. lem. 10.  
10.

c 1. 4.

Let AB be p. (a) Take the square numbers CE, so that CD — CE (ED) be not Q. (b) and  $CD : ED :: ABq : AFq$ . In a circle described the diameter AB (c) fit AF, and draw BF. Then AB, AF, are the lines required.

A

For  $ABq : AFq (d) :: CD : ED ; (e)$  therefore  $ABq$   
 $AFq$ . But  $AB$  is  $\dot{p}$ ;  $(f)$  therefore  $AF$  is also  $\dot{p}$ . But  
 because  $CD$  is  $Q$ ; and  $ED$  not  $Q$ ;  $(g)$  therefore shall  
 $AB$  be  $\sqsubset AF$ . Moreover by reason of the  $(b)$  right-  
 angle  $AFB$ , is  $ABq (k) = AFq + BFq$ ; therefore seeing  
 $ABq : AFq :: CD : ED$ , by conversion of proportion  
 all  $ABq : BFq :: CD : CE :: Q : Q (l)$  Therefore  
 $AB \sqsubset BF$ . Which was to be done.

d constr.  
 e 6. 10.  
 f sch. 12.  
 10.  
 g 9. 10.  
 h 31. 3.  
 k 47. 1.  
 l 9. 10.

In numbers, let there be  $AB, 6$ ;  $CD, 9$ ;  $CE, 4$ ; where-  
 fore  $ED, 5$ . Make  $9 : 5 :: 36 : (Q : 6.) AFq$ . then  $AFq$   
 will be 20; and consequently  $AF \sqrt{20}$ . Therefore  $BFq$   
 $36 - 20 = 16$ . Wherefore  $BF$  is 4.

PROP. XXXI. Plate IV. Fig. 36.

To find out two rational lines  $AB, AF$  commensurable  
 in power, so that the greater  $AB$  shall be in power more  
 than the less  $AF$  by the square of a right-line  $BF$  incom-  
 mensurable to it self in length.

Let  $AB$  be  $\dot{p}$ .  $(a)$  Take the square numbers  $CE, ED$ , so  
 that  $CD = CE + ED$  be not  $Q$ , and in the rest follow  
 the construction of the preced. prop. I say then the thing  
 required is done.

a 2 lem. 29.  
 10.

For, as above,  $AB, AF$ , are  $\dot{p} \sqsubset$  also  $ABq : BFq ::$   
 $CD : ED$ . therefore since  $CD$  is not  $Q$ ;  $AB, BF (b)$  shall  
 be  $\sqsubset$ . Which was to be done.

b 9. 10.

In numbers, let there be  $AB, 5$ .  $CD, 45$ .  $CE = 36$ .  
 $ED = 9$ . Make  $45 : 9 :: 25 (ABq) : 5 (AFq)$ ; therefore  
 $AF = \sqrt{5}$ . consequently  $BFq = 45 - 25 = 20$ . Wherefore  
 $BF = \sqrt{20}$ .

PROP. XXXII. Fig. 37.

To find out two-medial lines  $C, D$ , commensurable only in  
 power, comprehending a rational rectangle  $CD$ , so that the  
 greater  $C$  be more in power than the lesser  $D$  by the square  
 of a right-line commensurable in length to the greater.

$(a)$  Take  $A$  and  $B \dot{p} \sqsubset$ ; so as  $\sqrt{Aq} - Bq \sqsubset A (b)$   
 and make  $A : C :: C : B. (c)$  and  $A : B :: C : D$ . I say the  
 thing is done.

a 30. 10.  
 b 13. 6.  
 c 12. 6.  
 d constr.  
 e 22. 10.  
 f 17. 6.  
 g 10. 10.  
 h 24. 10.  
 k 17. 6.  
 l 15. 10.

For because  $A$  and  $(d) B$  are  $\dot{p} \sqsubset (e)$  therefore shall  $C$   
 $(\sqrt{AB})$  be  $\mu$ .  $(g)$  and thence also  $C \sqsubset D (b)$  there-  
 fore  $D$  is likewise  $\mu$ . Furthermore, whereas  $A : B (d) ::$   
 $C : D$ ; and by permutation  $A : C :: B : D :: C : B$ ; and  
 $A$  is  $\dot{p} \nu$ ; therefore shall  $CD (k) (Bq)$  be  $\dot{p} \nu$ . Lastly, be-  
 cause  $\sqrt{Aq} - Bq (d) \sqsubset A, (l)$  shall  $\sqrt{Cq} - Dq$  be  $\sqsubset$   
 therefore, &c. But if  $\sqrt{Aq} - Bq \sqsubset Aq$ , then shall  
 $\sqrt{Cq} - Dq$  be  $\sqsubset C$ . In

In numbers ; let there be  $A\ 8$ ,  $B\ \sqrt{48}$  ( $\sqrt{\ : 64}$  —  
therefore  $C = \sqrt{AB} = \sqrt{3072}$ . and  $D = \sqrt{17}$   
wherefore  $CD = \sqrt{5308416} = \sqrt{2304}$ .

## P R O P. XXXIII. Plate IV. Fig. 38.

To find out two medial-lines  $D$ ,  $E$ , commensurable  
power only, comprehending a medial rectangle  $DE$ , so  
the greater  $D$  shall be more in power than the less  $E$ ,  
the square of a right-line commensurable to the greater less

a 30. 10.

b lem. 21.

10.

c 13. 6.

d 12. 6.

e constr.

f sch. 12. 10.

g 22. 10.

h 10. 10.

k 24. 10.

l 22. 10.

m 16. 6.

n 15. 5.

(a) Take  $A$  and  $C$   $\rho$   $\sqcap$ , so that  $\sqrt{Aq - Cq} \sqcap A$ ,  
take also  $B$   $\sqcap$   $A$  and  $C$ , and make  $A : D$  (c) : :  $D : B$   
: :  $C : E$  then  $D$ , and  $E$  are the lines sought for.

For because  $A$  and  $C$  (e) are  $\rho$ , (e) and  $B$   $\sqcap$   $A$  and  
(f) therefore shall  $B$  be  $\rho$ , and  $D$  ( $\sqrt{AB}$ ) (g) shall be  
But because  $A : D : : C : E$ , therefore by permutation  
 $C : : D : E$ . Wherefore seeing  $A$   $\sqcap$   $C$ , therefore  $D$   
be  $\sqcap$   $E$ ; therefore  $E$  is  $\mu$ . Furthermore, (l) beca  
 $D : B : : C : E$ . and  $BC$  is  $\mu\nu$ ; also  $DE$ , equal to it, is  
Lastly, because  $A : C : : D : E$  (e) seeing  $\sqrt{Aq - Cq}$   
 $A$ ; therefore  $\sqrt{Dq - Eq} \sqcap D$ ; therefore, &c. But  
 $Aq - Cq \sqcap A$ . then  $\sqrt{Dq - Eq} \sqcap Eq$ .

In numbers, let there be  $A\ 8$ ,  $C\ \sqrt{48}$ ,  $B\ \sqrt{28}$ . Th  
 $D\ \sqrt{3072}$ . and  $E\ \sqrt{588}$ . wherefore  $D : E : :$   
 $\sqrt{3}$ . And  $DE = \sqrt{1344}$ .

## P R O P. XXXIV. Fig. 39.

To find out two right-lines  $AF$ ,  $BF$ , incommensurable  
power, whose squares added together make a rational square  
and the rectangle contained under them medial.

a 31 10.

b 10. 1.

c 28. 6.

d 12. 6.

e cor. 8. 6.

f 17. 6.

g 7. 5.

h 19. 10.

i 10. 10.

k 31. 3. &amp;

47. 1.

l constr.

m 1. ax. 1.

n 22. 10.

o 24. 10.

p sch. 22. 6.

(a) Let there be found  $AB$ ,  $CD$ ,  $\rho$   $\sqcap$ ; so that  $\sqrt{Aq - Cq} \sqcap AB$ ;  
 $-CDq \sqcap AB$ ; divide  $CD$  equally in  $G$ , (c) make  
rectangle  $AEB = GCq$ . Upon  $AB$ , the diameter, de  
a semicircle  $AFB$ , erect the perpendicular  $EF$ , and dr  
 $AF$ ,  $BF$ . These are the lines required.

For  $AE : BE$  (d) : :  $BA \times AE : AB \times BE$ . But  $BA$   
 $AE$  (e)  $= AFq$ ; and  $AB \times BE = FBq$ . (f) Therefore  $AE$   
 $EB : : AFq : FBq$ ; therefore since  $AE$  (g)  $\sqcap$   $EB$ ,  
 $AFq$  shall be  $\sqcap$   $FBq$ . Moreover  $ABq$  (h)  $AFq +$   
(l) is  $\rho\nu$ . Lastly  $EFq$  (l)  $= AEB$  (l)  $= CGq$ ; (m) theref  
 $EF = CG$ . Therefore  $DE \times AB = 2 EF \times AB$ . But  
 $\times AB$  (n) is  $\mu\nu$ . (o) Therefore  $AB \times ED$ ,  $EF$ , (p) or  $AE$   
 $FB$  is  $\rho\nu$ . Which was to be demonstrated.



The Explanation of the same by numbers.

Let AB be 6, CD  $\sqrt{12}$ ; then  $CG = \sqrt{\frac{12}{4}} = \sqrt{3}$ . But  $AE = 3 + \sqrt{6}$ . and  $EB = 3 - \sqrt{6}$ , whence AF shall be  $\sqrt{4 \cdot 18} + \sqrt{216}$ . and  $FB \sqrt{4 \cdot 18} - \sqrt{216}$ . Also  $AFq + FBq$  is 36, and  $AF \times FB = \sqrt{108}$ .

But AE is found in this manner. Because BA (6.) : AF : : AF : AE. therefore  $6 \cdot AE = AFq = AEq + 3 \cdot (EFq)$  therefore  $6 \cdot AE - AEq = 3$ . Put  $3 + e = AE$ . then  $18 + 6e - 9 - 6e - ee$ , that is,  $9 - ee = 3$ . or  $ee = 6$ . wherefore  $e = \sqrt{6}$ . and so  $AE = 3 + \sqrt{6}$ .

P R O P. XXXV. Plate IV. Fig. 4c.

To find out two right-lines AE, EB, incommensurable in power, whose squares added together make a medial figure, and the rectangle contained under them rational.

(a) Take AB and CF  $\mu \sqcap$ , so that  $AB \times CF$  be  $\rho\nu$ , and  $\sqrt{ABq - CFq} \sqcap AB$ , and let the rest be done as in the prec. prop. AE, EB are the lines required,

For, as it is shewn there,  $AEq \sqcap EBq$ . also  $ABq (AEq + EBq) \mu\nu$ . and lastly  $AB \times CF$  (b) is  $\rho\nu$ . (c) therefore also  $AB \times DE$ , that is,  $AE \times EB$ , is  $\rho\nu$ , therefore, &c.

P R O P. XXXVI. Fig. 41.

To find out two right-lines BA, AC, incommensurable in power, whose squares added together make a medial figure, and the rectangle also contained under them medial, and incommensurable to the figure composed of the squares.

(a) Take BC and EF  $\mu \sqcap$ , so that  $BC \times EF$  be  $\mu\nu$ . and  $\sqrt{BCq - EFq} \sqcap BC$ , and so forward, as in the prec. BA, AC, shall be the lines sought for.

For (as above)  $BAq \sqcap ACq$ , also  $BAq + ACq$  is  $\mu\nu$ . and  $BA \times AC$  is  $\mu\nu$ . Lastly, BC (b)  $\sqcap EF$ , and (c) so  $BC \sqcap EG$ ; likewise  $BC : EG$  (d) : :  $BCq : BC \times EG$  ( $BC \times AD$ , or  $BA \times AC$ ) (e) therefore  $BCq (ABq + ACq) \sqcap BA \times AC$ . therefore, &c.

Schol. Fig. 41.

To find out two medial-lines incommensurable both in length and power.

(a) Take

a 32. 10.

b const.

c sch. 12.

10.

d sch. 22. 6.

a 33. 10.

b const.

c 13. 10.

d 1. 6.

e 14. 10.

- a 36. 10.  
b 13. 6.  
c 17. 6.  
d 14. 10.

(a) Take  $BC \mu$ , and let  $BA \times AC$  be  $\mu\nu$ , and  $\square BC$  ( $BAq + ACq$ ) (b) make  $BA : H :: H : AC$ . then I say  $BC$  and  $H$  are  $\mu$ . For  $BC$  is  $\mu$ . (a) and  $BA \times AC$  ( $Hq$ ) is  $\mu\nu$ . wherefore  $H$  is also  $\mu$ . (d) Likewise  $BA \times AC$  ( $\square BCq$ ); therefore  $Hq$   $\square BCq$ . therefore, &c.

*Here begins the senaries of lines irrational by composition.*

P R O P. XXXVII. Plate IV. Fig. 42.

*If two rational-lines AB, BC, commensurable only in power are added together, the whole line AC is irrational, and is called a binomial-line, or of two names.*

- a hyp.  
b lem. 26.  
10.  
c 11. def. 10.

For because  $AB$  (a)  $\square BC$ , thence (b) shall  $ACq$  be  $\square ABq$ . But  $AB$  (a) is  $\rho$ . (c) therefore  $AC$  is  $\rho$ . Which was to be demonstrated.

P R O P. XXXVIII. Fig. 42.

*If two medial-lines AB, BC, commensurable in power only, are compounded; and contain a rational rectangle, the whole line AC is irrational and called a first bimedral-line.*

- a hyp.  
b lem. 26.  
10.  
c 11. def. 10.

For because  $AB$  (a)  $\square BC$ , (b) shall  $ACq$  be  $\square AB \times BC$ ,  $\rho\nu$ . (c) therefore  $AC$  is  $\rho$ . Which was to be demonstrated.

Lemma. Fig. 43.

*A rectangle AC, contained under a rational-line AB and an irrational-line BC, is irrational.*

- a hyp.  
b 21. 10.

For if the rectangle  $AC$  be affirmed  $\rho\nu$ , (a) then because  $AB$  is  $\rho$ , (b) the breadth  $BC$  shall be also  $\rho$ . against the Hyp.

P R O P. XXXIX. Fig. 44.

*If two medial-lines AB, BC, commensurable only in power, containing a medial rectangle, are compounded, the whole line AC shall be irrational, and is called a second bimedral-line.*

- a cor. 16. 6.  
b 47. 1. &  
11. 6.  
c hyp.  
d 16. 10.  
e 24. 10.

Upon the propounded line  $DE \rho$  (a) make the rectangle  $DF = ACq$ ; (b) and  $DG = ABq + BCq$ .

Because  $ABq$  (c)  $\square BCq$ , (d) therefore  $ABq + BCq$ , is  $DG$ ,  $\square ABq$ ; but  $ABq$  (e) is  $\mu\nu$ , (e) therefore  $DG$  is  $\mu\nu$ . But the rectangle  $ABC$  is taken  $\mu\nu$ , (e) and consequently

2 ABC

2 ABC (*f*) (HF) is  $\mu\nu$ . (*g*) therefore EG and GF are  $\rho$ . Also because DG (*b*)  $\sqsupset$  HF; and DG : HF :: (*k*) EG : GF; (*l*) therefore EG  $\sqsupset$  GF. (*m*) therefore the whole EF is  $\rho$  (*n*) wherefore the rectangle DF is  $\rho\nu$ . (*o*) therefore  $\sqrt{DF}$ , i. e. AC, is  $\rho$ . Which was to be demonstrated.

PROP. XL. Plate IV. Fig. 42.

If two right-lines AB, BC, commensurable only in power, are added together, making that which is composed of their squares rational, and the rectangle contained under them medial, the whole right-line AC is irrational, and is called a Major-line.

For whereas ABq + BCq (*a*) is  $\rho\nu$ , and (*b*)  $\sqsupset$  2 ABC (*c*)  $\mu\nu$ ; and so ACq (*d*) (ABq + BCq + 2 ABC) (*e*)  $\sqsupset$  ABq + BCq  $\rho\nu$ . (*f*) therefore shall AC be  $\rho$ . Which was to be demonstrated.

PROP. XLI. Fig. 45.

If two right-lines AC, CB, incommensurable in power, are added together, having that which is made of their squares added together medial, and the rectangle contained under them rational, the whole right-line AB shall be irrational, and is called a line containing in power a rational and a medial rectangle.

For 2 rectangles ACB (*a*)  $\rho\nu$ , (*b*)  $\sqsupset$  ACq + CBq (*c*)  $\mu\nu$ . (*d*) therefore 2 ACB (*d*)  $\sqsupset$  ABq. wherefore (*e*) AB is  $\rho$ . Which was to be demonstrated.

PROP. XLII. Fig. 46.

If two right-lines GH, HK, incommensurable in power, are added together, having both that which is composed of their squares medial, and the rectangle contained under them medial, and incommensurable to that which is composed of their squares, the whole right-line GK is irrational, and is called a line containing in power two medial figures.

Upon the propounded line FB  $\rho$  make the rectangles AF = GKq, and CF = GHq + HKq. Because GHq + HKq (CF) (*a*) is  $\mu\nu$ , the breadth CB (*b*) shall be  $\rho$ . Also because 2 rectangles GHK (*c*) (AD) (*a*) is  $\mu\nu$ , therefore AC (*b*) shall be  $\rho$ . Moreover because the rectangle AD (*a*)  $\sqsupset$  CF, (*d*) and AD : CF :: AC : CB, (*e*) thence shall AC be  $\sqsupset$  CB. (*f*) wherefore A is (*g*)  $\rho$ . therefore the rectangle AF. i. e. GKq is  $\rho\nu$ ; (*b*) and consequently GK is  $\rho$ . Which was to be demonstrated.

L 2

PROP.

f 4. 2.  
g 23. 10.  
h lem. 26.  
10.  
k 1. 6.  
l 10. 10.  
m 37. 10.  
n lem. 38.  
10.  
o 11. def.  
10.

a hyp.  
b sch. 12.  
10.  
c hyp. and  
24. 10.  
d 4. 2.  
e 17. 10.  
f 11. def. 10.

a hyp. and  
sch. 12. 10.  
b sch. 12.  
10.  
c hyp.  
d 17. 10.  
e 11. def.  
10.

a hyp.  
b 23. 10.  
c 4. 2.  
d 2. 6.  
e 10. 10.  
f 37. 10.  
g lem. 38.  
10.  
h 11. def.  
10.



## PROP. XLIII. Fig. 47.

A line of two names, or binominal, AB, can at one point only D be divided into its names, AD, DB.

If it be possible, let the binominal-line AB be divided at the point E, into other names AE, EB. It is manifest that the line AB is in both cases divided unequally, since  $AD \nmid DB$ , and  $AE \nmid EB$ .

a 37. 10.

b sch. 27.

10

c sch. 5. 2.

d sch. 12.

10.

e 27. 10.

Because the rectangles ADB, AEB (a) are  $\mu\alpha$ ; (a) and each of ADq, DBq, AEq, EBq is  $\rho\alpha$ ; (b) and so  $ADq + DBq$  (b) and  $AEq + EBq$  are also  $\rho\alpha$ . (b) therefore  $ADq + DBq = AEq + EBq$  (c) i. e.  $2 AEB = 2 ADB$  is  $\rho\nu$ . (d) therefore  $AEB = ADB$  is  $\rho\nu$ . therefore  $\mu\nu$  exceeds  $\mu\nu$  by  $\rho\nu$ . (e) Which is absurd.

## PROP. XLIV. Fig. 47.

A first bimedral-line AB is in one point only D divide into its names AD, DB.

a 38. 10.

b sch. 27.

10.

c sch. 5. 2.

d 27. 10.

Conceive AB to be divided into other names AE, EB, whereupon every one ADq, DBq, EBq, will be (a)  $\mu\alpha$ . and the rectangles ADB, AEB, and the doubles of them,  $\rho\alpha$ ; (b) therefore  $2 AEB = 2 ADB$ . (c) i. e.  $ADq + DBq = AEq + EBq$  is  $\rho\nu$ . (d) Which is absurd.

## PROP. XLV. Fig. 48.

A second bimedral-line AB, is divided into its names AC, CB, only at one point C.

a 39. 10.

b 16. ana

24. 10.

c 23. 10.

d 24. 10.

e 4. 2.

f lem. 26. 10

g 1. 6.

h 10. 10.

k 37. 10.

Suppose there were other names AD, DB. Upon the propounded line EF  $\rho$  make the rectangles  $EG = ABq$ , and  $EH = ACq + CBq$ , as also  $EK = ADq + DBq$ .

Because ACq, BCq (a) are  $\mu\alpha \nmid$ ; (b)  $ACq + CBq$  (EH) shall be  $\mu\nu$ . (c) therefore the breadth FH is  $\rho$ . (a) Moreover the rectangle ACB, (d) and so  $2 ACB$  (e) (IG) is  $\mu\nu$ . (c) therefore HG is also  $\rho$ . And since EH is (f)  $\nmid$  IG, (g) and  $EH : IG :: FH : HG$ . (b) therefore FH, HG shall be  $\nmid$  (h) therefore FG is a binomial, whose names are FH, HG. By the same reason FG is binomial, and the names of it FK, KG : contrary to the 43 of this Book.

PROP. XLVI. Plate IV. Fig. 47.

A major-line AB is at one point only D divided into its names AD, DB.

Imagine other names AE, EB, whereupon the rectangles ADB, AEB, (a)  $\mu\alpha$ . (a) and as well ADq + DBq, as AEq + EBq are  $\rho\alpha$ . (b) therefore ADq + DBq = AEq + EBq, (c) i. e. 2 AEB - 2 ADB is  $\rho\nu$ . (d) Which is impossible.

a 40. 10.  
b sch. 27.  
10.  
c sch. 5. 21.  
d 17. 10.

PROP. XLVII. Fig. 49.

A line AB containing in power a rational, and a medial-figure is divided at one point only D into its names AD, DB.

Conceive other names AE, EB, then both AEq + EBq, and ADq + DBq are  $\mu\alpha$ . (a) and the rectangles AEB, ADB are  $\rho\alpha$ . (b) therefore 2 AEB - 2 ADB, (c) i. e. ADq + DBq, AEq + EBq is  $\rho\nu$  (d) Which is absurd.

a 41. 10.  
b sch. 27.  
10.  
c sch. 5. 2.  
d 27. 10.

PROP. XLVIII. Fig. 48.

A line AB containing in power two medial-rectangles, is at one point only C divided into its names AC, CB.

If you would have AB to be divided into other names AD, DB, draw upon the line compounded EF  $\rho$  the rectangles EG = ABq, and EH = ACq + CBq, and EH = ADq + DBq. then because ACq + CBq, namely EH, (a) is  $\mu\nu$ , (b) the breadth FH shall be  $\rho$ . Also because 2 ACB, (c) that is, IG, is (a)  $\mu\nu$ , HG (b) shall be likewise  $\rho$ . Therefore, whereas EH (a)  $\sqsupset$  IG, and EH : IG (d) : : FH : HG, thence FH (e) shall be  $\sqsupset$  HG, (f) therefore FG is a binomial, and the names of it FH, HG. In like manner FK, KG shall be the names of it, against the 43. of this Book.

a 42. 10.  
b 23. 10.  
c 4. 2.  
d 1. 6.  
e 10. 10.  
f 37. 10.

Second Definitions.

**A** Rational-line being propounded, and the binomial divided into its names, the greater of whose names is more in power than the less by the square of a right-line commensurable to the greater in length; then

L 3

I. If

I. If the greater name be commensurable in length to the rational-line propounded, the whole line is called a binomial-line.

II. But if the lesser name be commensurable in length to the rational-line propounded, the whole line is called a second binomial.

III. If neither of the names be commensurable in length to the rational-line propounded, it is called a third binomial.

Furthermore, if the greater name be more in power than the less, by the square of a right-line incommensurable to the greater in length, then

IV. If the greater name be commensurable to the propounded rational-line in length, it is called a fourth binomial.

V. If the lesser name be so, a fifth.

VI. If neither, a sixth.

# PROP. XLIX.

A....4C.....5B

D\_\_\_\_\_

E\_\_\_\_\_F\_\_\_\_\_G

H\_\_\_\_\_

To find out a first binomial line, EG.

(a) Take AB, AC, square numbers, whose excess CB is not Q. let D be propounded.

(b) Take EF  $\square$  D, and (c) make AB : CB :: EFq : FGq. then EG shall be (a) first binomial.

For EF (d)  $\square$  D. (e) therefore EF is  $\rho$ . (f) also EFq  $\square$  FGq. (g) therefore FG is also  $\rho$ . likewise (d) because EFq : FGq :: AB : CB :: Q : not Q (b) therefore EF  $\square$  FG. Lastly, because by conversion of proportion, EFq : EFq - FGq :: AB : AC :: Q : Q thence EF (h) shall be  $\square$   $\sqrt{\text{EFq} - \text{FGq}}$ . (i) therefore EG is a first binomial. Which was to be done.

In numbers thus ; let there be D 8. EF 6. AB 9. CB 5. wherefore because 9 : 5 :: 36 : 20. therefore FG is  $\sqrt{20}$ . and consequently EG is  $= 6 + \sqrt{20}$ .

# PROP. L.

A....4C.....5B

D\_\_\_\_\_

E\_\_\_\_\_F\_\_\_\_\_G

H\_\_\_\_\_

To find out a second binomial line, EG.

Take AB and AC square numbers, the excess of which is CB not Q. Let the line D be

propounded  $\rho$ . take FG  $\square$  D, and make CB : AB :: FGq : EFq. then EG will be the line desired.

Prove it as the prec.

For



For  $FG \perp D$ . wherefore  $FG$  is  $\dot{p}$ . Also  $EFq \perp FGq$  therefore  $EF$  is  $\dot{p}$ . Likewise because  $FGq : EFq :: CB : AB :: \text{not } Q : Q$ , thence  $FG$  is  $\perp EF$ . Lastly, seeing  $CB : AB :: FGq : EFq$ , and inversely  $AB : CB :: EFq : FGq$ . therefore as in the foregoing Prop.  $EF \perp \sqrt{EFq - FGq}$ . (a) whereby  $EG$  is a second binomial. Which was to be done.

a 2 def. 48.  
10.

In numbers ; let there be  $D$  8,  $FG$  10,  $AB$  9,  $CB$  5 ; then  $EF$  is  $\sqrt{180}$ , wherefore  $EG$  is  $10 + \sqrt{180}$ .

PROP. LI.

To find out a third binomial line, DF.

A....4 C.....5 B  
L.....6.

(a) Take  $AB$ ,  $AC$ , square numbers, the excess of which  $CB$  is not  $Q$ , and let  $L$  be a number not  $Q$  next greater

G \_\_\_\_\_  
D \_\_\_\_\_ E \_\_\_\_\_ F  
H \_\_\_\_\_

a sch. 19.  
10.

than  $CB$ , viz. by a unit or two. Let  $G$  be the line compounded  $\dot{p}$  (b) Make  $L : AB :: Gq : DEq$ , (b) and  $AB : CB :: DEq : EFq$ , then  $DF$  shall be a third binomial.

b 3 lem. 10  
10

For because  $DEq (c) \perp Gq$ , (d)  $DE$  is  $\dot{p}$ . also  $Gq : DEq :: L : AB :: \text{not } Q : Q$ . (e) therefore  $G \perp DE$ . Likewise since  $DEq (e) \perp EFq$ , (d) also  $EF$  is  $\dot{p}$ . Moreover because  $DEq : EFq :: AB : CB :: Q : \text{not } Q$ , (f) is  $DE \perp EF$ . and since by constr. and equality  $Gq : EFq :: L : CB :: \text{not } Q : Q$ . (for (g)  $L$  and  $CB$  are not like plane numbers) (b) therefore shall  $G$  be also  $\perp EF$ . Lastly, as in the prec. prop.  $\sqrt{DEq - EFq} \perp DE$ . (k) therefore  $DF$  is a third binomial. Which was to be done.

c constr. 6.  
10.  
d sch. 12.  
10.  
e 6. 10.  
f 9. 10.  
g sch. 27. 8.  
h 9. 10.  
k 3 def. 48.  
10.

In numbers ; let there be  $AB$ , 9.  $CB$ , 5.  $L$ , 6.  $G$ , 8. then shall be  $DE \sqrt{96}$ , and  $EF \sqrt{4\frac{2}{3}}$ . wherefore  $DF = \sqrt{96} + \sqrt{4\frac{2}{3}}$ .

PROP. LII.

To find out a fourth binomial-line DF.

A...3 C.....6 B

(a) Take any square number  $AB$ , and divide it into  $AC$ ,  $CB$  not squares. Let  $G$  be the line compounded  $\dot{p}$ . (b) take  $DE \perp G$ , (c) and make  $AB : CB :: DEq : EFq$ , then  $DF$  shall be a fourth binomial.

G \_\_\_\_\_  
D \_\_\_\_\_ E \_\_\_\_\_ F  
H \_\_\_\_\_

a sch. 29.  
10.

b 2. lem. 1  
10.  
c 3 lem. 10.  
10.

L 4

For

d 9. 10.  
e 4. def. 48.  
10.

For, as in the 49 of this Book, DF may be shewn to be a binomial, and also because by constr. and conversion of proportion  $DEq : DEq - EFq :: AB : AC :: Q : \text{not } Q$ . (d) shall DE be  $\sqrt[4]{DEq - EFq}$  (e) therefore DF is a fourth binomial.

In numbers, let G be 8. DE, 6. then EF shall be  $\sqrt{24}$ . therefore DF is  $6 + \sqrt{24}$ .

## P R O P. LIII.

A ... 3 C ..... 6 F.

G —————  
D ————— E — F  
H —————

To find out a fifth binomial-line, DF.

Take any square number AB whose segments AC, CB are not Q. Let G be the line propounded

ed & take EF  $\sqrt[4]{G}$ . and make CB : AB :: EFq : DEq. then shall DF be a fifth binomial.

a 9. 10.  
b 5. def. 48.  
10.

For DF shall be a binomial as in the 50. of this Book, and because by construction, and inversion,  $DEq : EFq :: AB : CB$ , and so by conversion of proportion,  $DEq : DEq - EFq :: AB : AC :: Q : \text{not } Q$  (a) therefore shall DE be  $\sqrt[4]{DEq - EFq}$ . (b) therefore DF is a fifth binomial. Which was to be done.

## P R O P. LIV.

A ..... 5 C ..... 7 B

L ..... 9

G —————  
D ————— E — F  
H —————

To find out a sixth binomial-line.

Take AC, CB, prime numbers, so that AC + CB (AB) be not Q. take also any number square L. Let G be

a 3. lem. 10.  
10.

the line propounded &. (a) and make L : AB :: Gq : DEq, and AB : CB :: DEq : EFq. then DF shall be a sixth binomial.

b sch. 27. 8.  
c 9. 10.  
d 6 def. 48.  
10.

For DF may be demonstrated binomial as in the 51. of this Book, and also by reason that DE and E  $\sqrt[4]{FG}$ . Lastly likewise because by constr. and conversion of proportion  $DEq : DEq - EFq :: AB : AC :: \text{not } Q : Q$ . (For AB is prime to AC, (b) and so unlike to it) (c) therefore DE  $\sqrt[4]{DEq - EFq}$ , (d) therefore DF is a sixth binomial. Which was required.

In numbers, let there be G 6. DE  $\sqrt{48}$ . then EF shall be  $\sqrt{28}$ . wherefore DF is  $\sqrt{48} + \sqrt{28}$ .

Lemma.

Lemma. Plate V. Fig. 50.

Let AD be a rectangle, and the side thereof AC divided unequally in E; also let the lesser portion EC be equally divided in F. Upon the line AE (a) make the rectangle AGE = EFq, and from the points G, E, F, (b) draw GH, EI, FK, parallel to AB. (c) Let the square LM be made equal to the rectangle AH, and upon OMP produced the square MN = GI, and let the right-lines LOS, LQT, NRS, NPT, be produced.

a 28. 6.  
b 31. 1.  
c 14. 2.

I say 1. MS, MT, are rectangles. For by reason of the right-angles of the squares OMQ, RMP, (a) shall QMR be a right-line. (b) Therefore RMO, QMP, are right-angles, wherefore the parallelograms MS, MT, are rectangles.

a sch. 15. 1  
b 13. 1.

2. Hence it is plain that LS (c) = LT, and consequently that LN is a square.

c 2. 4. 1.

3. The rectangles SM, MT, EK, ED, are equal. For because the rectangle AGE (d) = EFq. (e) thence shall AG : EF :: EF : GE, (f) and so AH : EK :: EK : GI, that is by constr. LM : EK :: EK : MN. (g) but LM : SM :: SM : MN; therefore EK (h) = SM (k) = FD (l) = MT.

d hyp.  
e 17. 6.  
f 1. 6.  
g sch. 22. 6.  
h 9. 5.  
k 36. 1  
l 43. 1.  
m 2 ax. 1.  
n 16. 10.  
o 18. and  
16. 10.  
p 10. 10.  
q 14. 10.  
r 10. 10.

4. Hence LN (m) = AD.

5. Because EC is equally divided in F, (n) it is plain that EF, FC, EC are  $\square$ .

6. If AE  $\square$  EC, and AE  $\square$   $\sqrt{AEq - ECq}$ , (o) then shall AG, GE, AE, be  $\square$ . also because AG : GE :: AH : GI, (p) therefore shall AH, GI, i. e. LM, MN, be  $\square$ . Likewise

7. OM  $\square$  MP. For by the Hyp. AE  $\square$  EC (q) therefore EC  $\square$  GE. (q) wherefore EF  $\square$  GE. but EF : GE :: EK : GI (r) therefore EK  $\square$  GI that is, SM  $\square$  MN. But SM : MN :: OM : MP. (r) therefore OM  $\square$  MP.

8. If AE be supposed  $\square$   $\sqrt{AEq - ECq}$ , it is apparent that AG, GE, AE, are  $\square$  whence (f) LM  $\square$  MN. for AG : GE :: AH : GI :: LM : MN.

f 19. and  
17. 19.

These being well considered, we shall easily dispatch the six following Propositions.

P R O P.



## PROP. LV. Plate IV. Fig. 50.

If a space AD be contained under a rational-line AB, and a first binomial-line AC (AE + EC) the right-line OP which containeth that space in power is irrational, and called a binomial-line.

a hyp. and

lem. 54. 10.

b hyp.

c sch. 12. 10.

d 20. 10.

e lem. 54. 10.

f 37. 10.

All that being supposed which is described and demonstrated in the foregoing Lemma, it is manifest that the right-line OP containeth in power the space AD. (a) Likewise AG, GE, AE, are  $\perp$ . therefore seeing AE, (b) is  $\dot{p} \perp$  AB, (c) shall also AG and GE be  $\dot{p} \perp$  AB. (d) Therefore the rectangles AH, GI, that is, the squares LM, MN, are  $\dot{p}a$ . therefore OM, MP are  $\dot{p} (e) \perp$ , (f) and consequently OP is a binomial. Which was to be demonstrated.

In numbers, let there be AB 5, AC 4 +  $\sqrt{12}$ . wherefore the rectangle AD = 20 +  $\sqrt{300}$  = to the square LN. Therefore OP is  $\sqrt{15} + \sqrt{5}$ . namely a sixth binomial.

## PROP. LVI. Fig. 50.

If a space AD be comprehended under a rational-line AB, and a second binomial AC (AE + EC) the right-line OP, which containeth that space AD in power, is irrational, and called a first medial-line.

a hyp. and

lem. 54. 10.

b hyp.

c sch. 12. 10.

d 12. 10.

e lem. 54. 10.

f hyp. 12.

10.

g 20. 10.

h 38. 10.

The foresaid Lemma of the 54 of this Book being again supposed, then shall OP be =  $\sqrt{AD}$ . (a) also AE, AG, GE, are  $\perp$ . therefore since AE (b) is  $\dot{p} \perp$  AB, likewise AG, GE (c) shall be  $\dot{p} \perp$  AB, therefore the rectangles AH, GI, i. e. OMq, MPq. (d) are  $\mu a$ , (e) Moreover OM  $\perp$  MP. Lastly, EF  $\perp$  EC, and EC (f)  $\perp$  AB. (g) Wherefore EK, i. e. SM, or OMP, is  $\dot{p}v$ , (h) Consequently OP is a first bimedral. Which was to be dem.

In numbers, let there be AB 5, and AC  $\sqrt{48} + 6$ . then the rectangle AD =  $\sqrt{1200} + 30$  = OPq. therefore OP is  $\sqrt{675} + \sqrt{75}$ , viz. a first bimedral.

## PROP. LVII. Fig. 50.

If a space AD be contained under a rational-line AB, and a third binomial-line AC (AE + EC) the right-line OP which containeth in power the space AD, is irrational, and called a Second bimedral-line.

As above,  $OPq = AD$ . Also the rectangles  $AH, GI$ , that is,  $OMq, MPq$  are  $\mu\alpha$  (a) Likewise  $EK$ , or  $OMP$  is  $\mu\nu$ . (b) Therefore  $OP$  is a second binomial. In numbers, let there be  $AB\ 5$ .  $AC\ \sqrt{32} + \sqrt{24}$ . wherefore  $AD$  is  $\sqrt{800} + \sqrt{600} = OPq$ . and so  $OP$  is  $\sqrt{450} + \sqrt{50}$ , that is, a second binomial.

a hyp. and  
22. 10.  
b 39. 10.

PROP. LVIII. Plate IV. Fig. 50.

If a space  $AD$  be comprehended under a rational-line  $AB$ , and a fourth binomial  $AC$  ( $AE + EC$ ) the right-line  $OP$  containing the space  $AD$  in power, is that irrational-line which is called a Major-line.

For again,  $OMq$  (a)  $\sqsubset MPq$ ; and the rectangle  $AI$ , e.  $OMq + MPq$  (b) is  $\rho\nu$ . (c) also  $EK$  or  $OMP$  is  $\mu\nu$ . (d) therefore  $OP$  ( $\sqrt{AD}$ ) is a Major line. Which was to be demonstrated.

In numbers, let there be  $AB\ 5$ . and  $AC\ 4 + \sqrt{8}$ . then the rectangle  $AD$  is  $20 + \sqrt{200}$ . wherefore  $OP$  is  $\sqrt{20} + \sqrt{200}$ .

a lem. 54.  
10.  
b hyp. and  
20. 10.  
c hyp. and  
22. 10.  
d 40. 10.

PROP. LIX.

If a space  $AD$  be contained under a rational-line  $AB$ , and a fifth binomial  $AC$ , the right-line  $OP$  which containeth the space  $AD$  in power, is that irrational-line, which is a line containing a rational and a medial rectangle in power.

Again  $OMP \sqsubset MPq$ . and the rectangle  $AI$  or  $OMq + MPq$  is  $\mu\nu$ . (a) Likewise the rectangle  $EK$  or  $OMP$  is  $\rho\nu$ . (b) therefore  $OP$  ( $\sqrt{AD}$ ) contains in power  $\rho\nu$  and  $\mu\nu$ . Which was to be demonstrated.

In numbers, let there be  $AB\ 5$ , and  $AC\ 2 + \sqrt{8}$ ; then the rectangle  $AD = 10 + \sqrt{200} = OPq$ . Wherefore  $OP$  is  $\sqrt{10} + \sqrt{200}$ .

a as in the  
prec.  
b 41. 10.

PROP. LX.

If a space  $AD$  be contained under a rational-line  $AB$  and a sixth binomial  $AC$  ( $AE + EC$ ) the line  $OP$  containing the space  $AD$  in power is irrational, which containeth in power two medial-rectangles.

As often before,  $OMq \sqsubset MPq$ . and  $OMq + MPq$  is  $\mu\nu$ . and also the rectangle ( $EK$ )  $OMP$  is  $\rho\nu$ . (a) therefore  $OP = \sqrt{AD}$  contains in power 2  $\mu\alpha$ . Which was to be demonstrated.

In numbers, let there be  $AB\ 5$ ,  $AC\ \sqrt{12} + \sqrt{8}$ . wherefore the rectangle  $AD$  or  $OPq$  is  $\sqrt{300} + \sqrt{200}$ . and so  $OP$  is  $\sqrt{300} + \sqrt{200}$ , Lemma

c 42. 10.

## Lemma. Plate IV. Fig. 51.

Let a right-line AB be unequally divided in C, and AC be the greater segment, and upon some line DE apply the rectangles  $DF = ABq$ , and  $DH = ACq$ , and  $IK = CBq$ , and let LG, be divided equally in M, and also MD drawn parallel to GF.

a 4. 2. and

3. ax. 1.

b 7. 2.

c 1. 6.

d 16. 10.

e lem. 26.

10.

f 10. 10.

g 1. 6.

h 17. 6.

k hyp.

l 10. 10.

m 18. 10.

n 19. 10.

I say, 1. The rectangle ACB is  $= LN$  or  $MF$ . (a) For  $2 ACB = LF$ .

2.  $DL \sqsubset LG$ . for  $DK (ACq + CBq)$  (b)  $\sqsubset LF (ACB)$  therefore since  $DK, LF$  are of equal altitude, (c)  $DL$  shall be  $\sqsubset LG$ .

3. If  $AC \sqsupset CB$ , (d) then shall the rectangle  $DK \sqsupset ACq$  and  $CBq$ .

4. Also  $DL \sqsupset LG$ . For  $ACq + CBq$  (e)  $\sqsupset 2 ACq$ . i. e.  $DK \sqsupset LF$ , but  $DK : LF$  (e)  $:: DL : LG$ , (f) therefore  $DL \sqsupset LG$ .

5. Moreover  $DL \sqsupset \sqrt{DLq - LGq}$ . For  $ACq : AC$  (g)  $:: ACB : CBq$ . that is  $DH : LN :: IK$ . (c)  $LN$  wherefore  $DL : LM :: LM : IL$ ; (b) therefore  $DI \times IL = LMq$ . therefore seeing  $ACq$  (k)  $\sqsupset CBq$ , that is,  $DH \sqsupset IK$ , and (l) so  $DI \sqsupset IL$ , (m) shall  $DL$  be  $\sqsupset \sqrt{DLq - LGq}$ . Which was to be demonstrated.

6. But if  $ACq$  be put  $\sqsupset CBq$ , (n) then shall  $DL$  be  $\sqrt{DLq - LGq}$ .

This Lemma is preparatory to the six following Propositions.

## P R O P. LXI.

The square of a binomial-line  $(AC + CB)$  applied to a rational-line  $DE$ , makes the latitude  $DG$  a first binomial-line.

a hyp.

b lem. 60.

10.

c sch. 12. 10.

d 21. 10.

e 22. and

24. 10.

f 23. 10.

g 13. 10.

h lem. 60.

10.

k 1 def. 48.

10.

Those things being supposed, which are described and demonstrated in the preceding Lemma, because  $AC, CB$  (a) are  $\dot{p} \sqsupset$ , (b) the rectangle  $DK$  shall be  $\dot{p} \sqsupset$   $ACq$ ; (c) and so  $DK$  is  $\dot{p} \sqsupset$ ; (d) therefore  $DL \sqsupset DE$ . But the rectangle  $ACB$ , and so  $2 ACB (LF)$  (e) is  $\dot{p} \sqsupset$  (f) therefore the latitude  $LG$  is  $\dot{p} \sqsupset DE$ , (g) therefore also  $DL \sqsupset LG$  also  $DL \sqsupset \sqrt{DLq - LGq}$ . From whence (k) it follows that  $DG$  is a first binomial-line. Which was to be demonstrated.

P R O



PROP. LXII.

The square of a first bimedial-line ( $AC+CB$ ) being applied to a rational-line  $DE$ , makes the latitude  $DG$  a second binomial-line.

The aforesaid Lemma being again supposed; The rectangle  $DK \sqsupset ACq$ ; (a) therefore  $DK$  is  $\mu\nu$ ; (b) therefore the latitude  $DL$  is  $\rho \sqsupset DE$ . But because the rectangle  $ACB$ , and so  $LF$  ( $2 ACB$ ) (c) is  $\rho\nu$ , (d) shall  $LG$  be  $\rho \sqsupset DE$ ; (e) therefore  $DL, LG$  are  $\sqsupset$ ; (f) also  $DL \sqsupset \sqrt{DLq - LGq}$ ; (g) from whence it is clear that  $DG$  is a second binomial. Which was to be demonstrated.

PROP. LXIII.

The square of a second bimedial-line ( $AC+CB$ ) applied to a rational-line  $DE$  makes the breadth  $DG$  a third binomial-line.

As in the prec.  $DL$  is  $\rho \sqsupset DE$ . Furthermore because the rectangle  $ACB$ , and so  $LF$  ( $2 ACB$ ) (a) is  $\mu\nu$ ; (b) therefore shall  $LG$  be  $\rho \sqsupset DE$ . (c) Moreover  $DL \sqsupset LG$ . and also  $DL \sqsupset \sqrt{DLq - LGq}$ ; (d) therefore  $DG$  is a third binomial, Which was to be demonstrated.

PROP. LXIV.

The square of a Major-line ( $AC+CB$ ) applied to a rational-line  $DE$ , makes the breadth  $DG$  a fourth binomial-line.

Again  $ACq + CBq$ . i. e.  $DK$  (a) is  $\rho\nu$ , (b) therefore  $DL$  is  $\rho \sqsupset DE$ , also  $ACB$ , and so  $LF$  ( $2 ACB$ ) (c) is  $\mu\nu$ ; (d) therefore  $LG$  is  $\rho \sqsupset DE$ , (e) and consequently  $DL \sqsupset LG$ . Lastly, because  $AC \sqsupset BC$ , (f) shall  $DL \sqsupset \sqrt{DLq - LGq}$ , (g) whence  $DG$  is a fourth binomial. Which was to be demonstrated.

PROP. LXV.

The square of a line containing in power a rational and a medial rectangle ( $AC+CB$ ) applied to a rational-line  $DE$  makes the latitude  $DG$  a fifth binomial.

Again,  $DK$  is  $\mu\nu$ ; (a) therefore  $DL$  is  $\rho \sqsupset DE$ ; also  $LF$  is  $\rho\nu$ ; (b) therefore  $LG$  is  $\rho \sqsupset DE$ ; (c) therefore  $DL \sqsupset LG$ ; (d) likewise  $DL \sqsupset \sqrt{DLq - LGq}$ ; (e) and so by consequence  $DG$  is a fifth binomial. Which was to be demonstrated.

PROP.

a 24. 10.  
b 23. 10.  
c hyp. and  
sch. 22. 10.  
d 21. 10.  
e 13. 10.  
f lem. 60.  
10.  
g 2. def. 48.  
10.

a hyp. and  
24. 10.  
b 23. 10.  
c lem. 60.  
10.  
d 3. def. 48.  
10.

a hyp. and  
sch. 12. 10.  
b 21. 10.  
c hyp. and  
24. 10.  
d 23. 10.  
e 13. 10.  
f lem. 60.  
10.

g 4. def. 48.  
10.  
a 23. 10.  
b 21. 10.  
c 13. 10.  
d lem. 60.  
10.  
e 5. def. 48.  
10.

## P R O P. LXVI.

The square of a line containing in power two medial tangles ( $AB \perp CB$ ) applied to a rational-line DE, makes the latitude DG a sixth binomial-line.

a hyp.

b 14. 10.

c 1. 6.

d 10. 10.

e lem. 60.

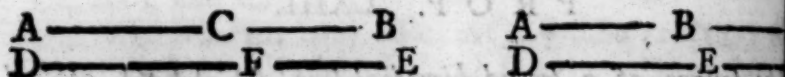
10.

f 6. def. 48.

10.

As before, DL and LG are  $\dot{p}$   $\perp$  DE. But because  $ACq \perp CBq$  (DK) (a)  $\perp$  ACB, (b) and so DK  $\perp$  (2 ACB) and also DK : LF (c) :: DL : LG. (d) therefore shall DL be  $\perp$  LG. (e) Lastly DL  $\perp \sqrt{DLq - LG}$  (f) by which it appears that DG is a sixth binomial.

Lemma.



Let AB, DE be  $\perp$ . and make  $AB : DE :: AC : DF$ . I say 1.  $AC \perp DF$ , as appears by 10. 10. also  $CB \perp FE$ ; (a) because  $AB : DE :: CB : FE$ .

a 19. 5.

2.  $AC : CB :: DF : FE$ . For  $AC : DF :: AB : DE :: CB : FE$ ; therefore, by permutation,  $AC : CB :: DF : FE$ .

b 1. 6.

c before

d 10. 10.

3. The Rectangle  $ACB \perp DFE$ . For  $ACq : ACB :: AC : CB$  (c) ::  $DF : EF :: DFq : DFE$ , wherefore permutation  $ACq : DFq :: ACB : DFE$ , therefore shall  $ACq \perp DFq$ , (d) shall  $ACB$  be  $\perp$  DFE.

e 22. 6.

f 10. 10.

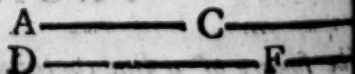
g 10. 10.

4.  $ACq \perp CBq \perp DFq \perp FEq$ . For because  $ACq : CBq :: DFq : FEq$  (e) ::  $DFq : FEq$ ; therefore by compound  $ACq \perp CBq : CBq :: DFq \perp FEq : FEq$ , therefore shall  $CBq \perp FEq$ , (f) shall also  $ACq \perp CBq$  be  $\perp$  DFE.

5. Hence, If  $AC \perp$  or  $\perp CB$ , (g) then likewise shall DE be  $\perp$  or  $\perp EF$ .

## P R O P. LXVII.

A line DE, commensurable in length to a binomial line ( $AC \perp CB$ ) is itself a binomial-line, and of the same order.



a lem. 66.

10.

b hyp.

c lem. 66.

10. and

fch. 12. 10.

d 15. 10.

Make  $AB : DE :: AC : DF$ ; (a) then are AC, (a) and CB, FE  $\perp$ ; whence since AC and CB are  $\dot{p}$   $\perp$ , (c) thence, DF, FE  $\dot{p}$   $\perp$ ; therefore DF binomial. But because  $AC : CB$  (a) ::  $DE : FE$ .  $AC \perp$  or  $\perp \sqrt{ACq - BCq}$ , (d) then in like manner

DF  $\square$  or  $\square \vee$  DFq — FEq; also if AC  $\square$  or  $\square \vee$  propounded, (e) then shall DF be  $\square$  or  $\square \vee$  propounded. But if CB  $\square$  or  $\square \vee$  likewise FE  $\square$  or  $\square \vee$ . If both AC, CB,  $\square \vee$ , (f) then also both DF, FE,  $\square \vee$ . (g) That is, whatsoever binomial AB is, DE shall be of the same order. *Which was to be demonstrated.*

e 12. 10.  
and 14. 10.  
f by def. 48  
10.  
g 14. 10.

PROP. LXVIII.

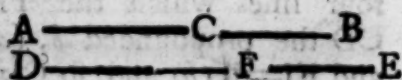
A line DE commensurable in length to a binomial-line (AC + CB) is also a binomial-line, and of the same order.

Make AB : DE :: AC : DF; (b) therefore AC  $\square$  + CB  $\square$  + EF; therefore seeing AC and CB (c) are  $\square$  +  $\square \vee$  (d) also DF and FE shall be  $\mu$ ; and because AC (e)  $\square$  + CB, (e) therefore FD  $\square$  + FE. (f) therefore DE is  $2\mu$ . Therefore if the rectangle ACB is  $\rho$ ; because DFE  $\square$  + ACB, (g) likewise DFE is  $\rho \vee$ ; and if that be  $\mu \vee$ , this shall be  $\mu \vee$  too. (k) That is, whether AB be 1 med, or 2 bimed, DF shall be of the same order. *Which was to be dem.*

a 12. 6.  
b lem. 66.  
10.  
c hyp.  
d 24. 10.  
e 10. 10.  
f 38. 10.  
g sch. 12.  
10.  
h 24. 10.  
k 38. or 39  
10.

PROP. LXIX.

A line DE commensurable to a Major-line (AC + CB) is itself a Major-line.



Make AB : DE :: AC : DF. Because AC (a)  $\square$  + CB (b) thence DF  $\square$  + FE; also ACq + CBq (a) is  $\rho \vee$ ; so because DFq + FEq (b)  $\square$  + ACq + CBq; (c) DFq + FEq is  $\rho \vee$ ; lastly, the rectangle ACB (a) is  $\rho$ ; (d) therefore the rectangle DFE is  $\mu \vee$  (because DFE is (b)  $\square$  + ACB) (e) wherefore DE is a Major-line. *Which was to be demonstrated.*

a hyp.  
b lem. 66.  
10.  
c sch. 12. 10.  
d 24. 10.  
e 40. 10.

PROP. LXX.

A line DE commensurable to a line containing in power a rational and a medial-rectangle (AC + CB) is a line containing in power a rational and a medial-rectangle.

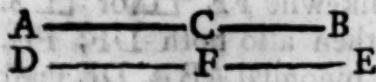
Again make AB : DE :: AC : DF. Because AC (a)  $\square$  + CB, (b) also DF  $\square$  + FE; likewise because ACq + CBq (a) is  $\mu \vee$  (c) therefore DFq + FEq shall be  $\mu \vee$ ; because the rectangle ACB (a) is  $\rho \vee$ , (d) also DFE is  $\rho \vee$ . Therefore DE contains in power  $\rho \vee$  and  $\mu \vee$ . *Which was to be dem.*

a hyp.  
b lem. 66.  
10.  
c 24. 10.  
d sch. 12. 10.

PROP.



## P R O P. LXXI.



A line DE commensurable to  
a line containing two medial-rec-  
tangles in power ( $AC + CB$ ) is

also a line containing in power two medial-rectangles.

a hyp.

b lem. 66.

10.

c 24. 10.

d 24. 10.

e 14. 10.

f 42. 10.

Divide DE, as in the prec. Because ACq (a)  $\square$  CBq, (b) thence shall DFq be  $\square$  FFq; also because ACq + CBq (a) is  $\mu\nu$ , (c) shall DFq + FEq be also  $\mu\nu$ . And in like manner because ACB (a) is  $\mu\nu$ , (d) also DFE is  $\mu\nu$ . Lastly, because ACq + CBq  $\square$  ACB, (e) shall DFq + FEq be  $\square$  DFE. (f) From whence it follows that DE contains in power 2  $\mu\alpha$ . Which was to be demonstrated.

## P R O P. LXXII. Plate V. Fig. 1.

If a rational-rectangle A and a medial B, are compounded, four irrational-lines will be made; either a binomial, or a first bimedial or a Major, or a line containing in power a rational and a medial-rectangle.

Namely, If  $Hq = A + B$ , then H shall be one of the four lines which the Theorem mentions. For upon CD the propounded  $\dot{p}$ , (a) make the rectangle  $CE = A$ , and  $FI = B$  (b) and so  $CI = Hq$ . Whereas then is A  $\dot{p}$ , likewise, CE is  $\dot{p}\nu$ . (c) therefore the latitude CP is  $\dot{p} \square$  CD, and because B is  $\mu\nu$ , also FI shall be  $\mu\nu$ ; (d) therefore FK is  $\dot{p} \square$  CD, (e) therefore CF, FK are  $\dot{p} \square$ , and so the whole CK (f) is binom. wherefore is A  $\square$  B, i.e. CE  $\square$  FI, (g) then CF  $\square$  FK. therefore if CF  $\square$   $\sqrt{CFq - FKq}$ ; (h) likewise CK shall be a 1 bin. and consequently H  $= \sqrt{CI}$  (k) is a bin. If CF be supposed  $\square \sqrt{CFq - FKq}$ , (l) then shall CK be a 4 bin. wherefore H ( $\sqrt{CI}$ ) (m) is a major-line. But if A  $\square$  B, (n) then shall CF be  $\square$  FK. consequently if FK  $\square \sqrt{FKq - CFq}$ , (o) then shall CK be a 2. bin. (p) wherefore H is a first 2  $\mu$ . Lastly if FK  $\square \sqrt{FKq - CFq}$ , (q) then CK shall be a fifth binom. (q) whence H shall contain in power  $\dot{p}\nu$  and  $\mu\nu$ . Which was to be demonstrated.

a cor. 16 6.

b 2. ax. 1.

c 21. 10.

d 23. 10.

e 33. 10.

f 37. 10.

g 1. 6.

h 1. def. 48.

10.

k 55. 10.

l 4. def. 48.

10.

m 58. 10.

n 2. def. 48.

10.

o 56. 10.

p 5. def. 48.

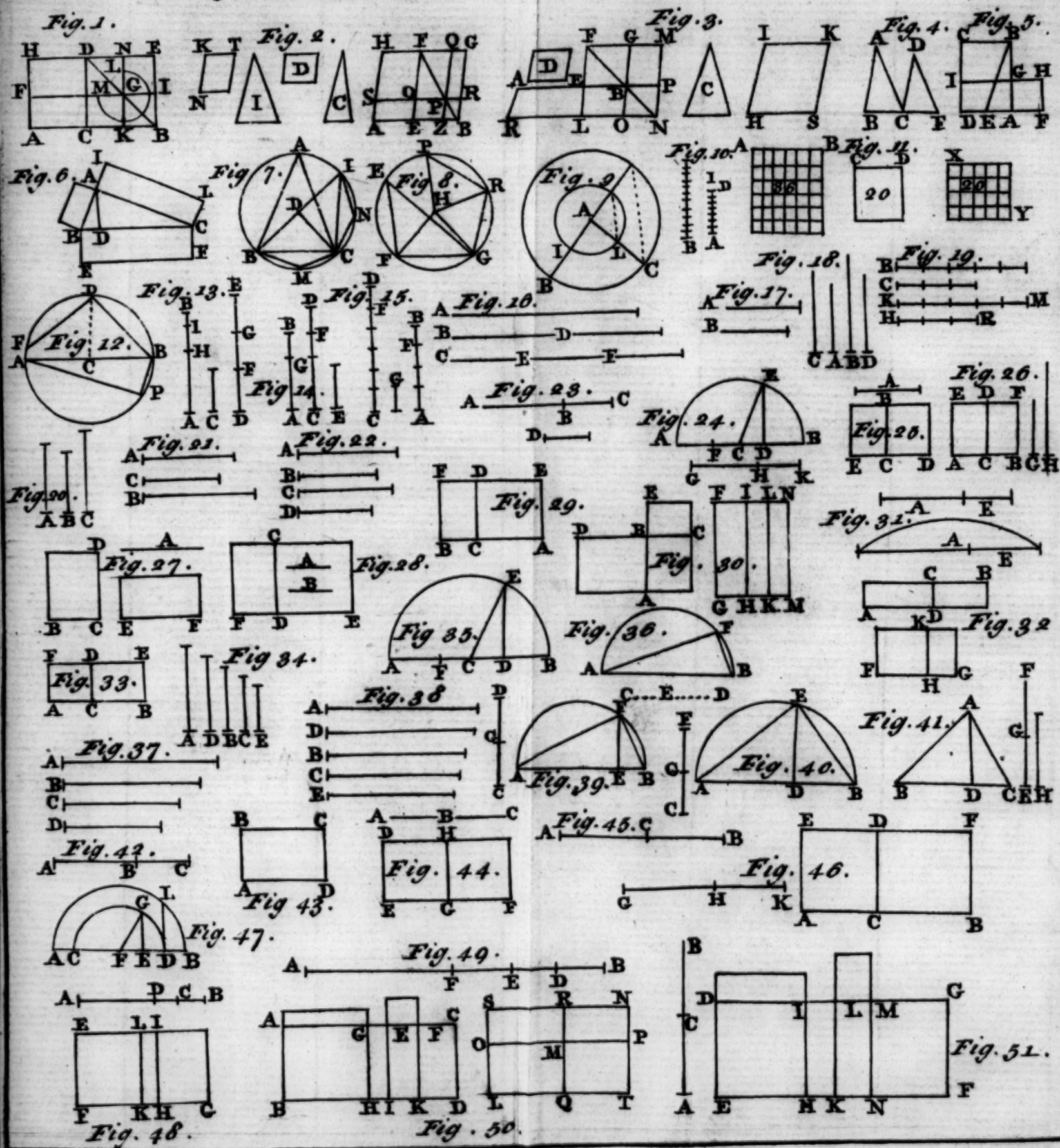
10.

q 59. 10.

## P R O P. LXXIII. Fig. 1.

If two medial rectangles, A, B, incommensurable to one another are composed, two remaining irrational-lines are made, either a second bimedial, or a line containing in power two medial rectangles.







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As H containing in power  $A + B$  is one of the said irrational lines. For upon CD propounded  $\rho$  make the rectangle  $CE = A$ , and  $FI = B$  whence  $Hq = CI$ . Therefore because CE and FI (a) are  $\mu\alpha$ , (b) the latitudes CF, FK, shall be  $\rho \sqcap CD$ . Also because CE (a)  $\sqcap$  FI, and  $CE : FI$  (c)  $:: CF : FK$ , (d) therefore  $CF \sqcap FK$ , (e) therefore CK is a bin. namely, if  $CF \sqcap \sqrt{CFq - FKq}$ , whence  $H = \sqrt{CI}$  (f) shall be  $2\mu$ . But if  $CF \sqcap \sqrt{CFq - FKq}$ , (g) then CK shall be a 6 binom. (b) and consequently H contains in power  $2\mu\alpha$ . Which was to be demonstrated.

a hyp  
b 23. 10.  
c 1. 6.  
d 10. 10.  
e 3. def. 48.  
10  
f 57. 10.  
g 6 def. 48.  
10.  
h 60. 10.

Here begins the Senaires of lines irrational by Subtraction.

PROP. LXXIV.

If from a rational-line DF a D — E — F rational-line DE, commensurable in power only to the whole DF, be taken away, the residue EF is irrational, and is called an Apotome or residual-line.

For EFq (a)  $\sqcap$  DEq; (b) but DEq is  $\rho\gamma$ ; (c) therefore EF is  $\rho$ . Which was to be demonstrated.

In numbers; let there be DF, 2. DE,  $\sqrt{3}$ . then EF shall be  $2 - \sqrt{3}$ .

a lem. 26.  
10.  
b hyp.  
c 10 & 11.  
def. 10,

PROP. LXXV.

If from a medial-line DF a D — E — F medial-line DE commensurable only in power to the whole DF, and comprehending with the whole DF a rational rectangle, be taken away, the remainder EF is irrational, and is called a first residual-line of a medial.

For EFq (a)  $\sqcap$  to the rectangle FDE. therefore seeing FDE (b) is  $\rho\gamma$ . (c) EF shall be  $\rho$ . Which was to be demonstrated.

In numbers, let DF be  $\nu\sqrt{54}$ , and DE  $\nu\sqrt{24}$ , therefore EF is  $\nu\sqrt{54} - \nu\sqrt{24}$ .

a sch. 26.  
10.  
b hyp.  
c 20. and  
11 def. 10.

PROP. LXXVI.

If from a medial-line DF, a D — E — F medial-line DE, be taken away being commensurable only in power to the whole DF, and com-  
M prehending

prehending together with the whole line DF a medial rectangle, the remainder EF is irrational, and is called a second residual of a medial-line.

a hyp. b 16. 10. c 24. 10. d cor. 7. 2. e 27. 10. Because DFq and DEq (a) are  $\mu\alpha$   $\square$ , (b) therefore shall DFq + DEq be  $\square$  DEq; (c) wherefore DFq + DEq is  $\mu\nu$ . Also the rectangle FDE, and so 2 FDE, (a) is  $\mu\nu$ , therefore EFq (d) (DFq + DEq - 2 FDE) (e) is  $\rho$ . wherefore EF is  $\rho$ . Which was to be demonstrated.

In numbers, let DF be  $\sqrt{18}$ . and DE  $\sqrt{8}$ . then EF  $\sqrt{18} - \sqrt{8}$ .

## P R O P. LXXVII.

A ——— B ——— C If from a right-line AC be taken away a right-line AB being incommensurable in power to the whole line AC, and making with the whole AC that which is composed of their squares rational, and the rectangle contained under them medial, the remainder BC is irrational, and is called a Minor-line.

a hyp. b sch. 12. 10. c 7. 2. d 17. 10. e 11. def. 10. For ACq + ABq (a) it  $\rho\nu$ . but the rectangle ACB (a) is  $\mu\nu$ ; (b) therefore 2 CAB  $\square$  ACq + ABq (c) (2 CAB + BCq.) (d) therefore ACq + ABq  $\square$  BCq. (e) therefore BC is  $\rho$ . Which was to be demonstrated.

In numbers, let AC be  $\sqrt{18} + \sqrt{108}$ ; AB  $\sqrt{18} - \sqrt{108}$ . then BC is  $\sqrt{18} + \sqrt{108} - \sqrt{18} - \sqrt{108}$ .

## P R O P. LXXVIII.

D ——— E ——— F If from a right-line DF be taken away a right-line DE, being incommensurable in power to the whole line DF, and with the whole DF making that which is composed of their squares medial, and the rectangle contained under the same lines rational, the line remaining EF is irrational, and is called a line making a whole space medial with a rational space.

a hyp. and sch. 12. 10. b hyp. c sch. 12. 10. d 7. 2. e sch. 12. 10. and 11. def. 10. For 2 FDE (a) is  $\rho\nu$ , (b) and DFq + DEq is  $\mu\nu$  (c) therefore 2 FDE  $\square$  DFq + DEq (d) (2 FDE + EFq) (e) therefore EF is  $\rho$ . Which was to be demonstrated.

In numbers, let DF be  $\sqrt{216} + \sqrt{72}$ ; DE  $\sqrt{216} - \sqrt{72}$ . therefore EF is  $\sqrt{216} + \sqrt{72} - \sqrt{216} - \sqrt{72}$ .

P R O P.



PROP. LXXIX.

$D \text{---} E \text{---} F$  If from a right-line  $DF$  be taken away a right-line  $DE$ , incommensurable in power to the whole  $DF$ , and which together with the whole makes that which is composed of their squares medial, and the rectangle contained under them also medial and incommensurable to that which is composed of their squares, the remainder is irrational, and is called a line making a whole space medial with a medial space.

For  $2 FDE$ , and  $FDq + DEq$  (a) are  $\mu$ ; (b) therefore  $EFq$  (c) ( $DFq + DEq - 2 FDE$ ) is  $\rho$ . (d) and so consequently  $EF$  is  $\rho$ . Which was to be demonstrated.

In numbers; let  $DF$  be  $\sqrt{180} + \sqrt{60}$ .  $DE$   $\sqrt{180} - \sqrt{60}$ , then  $EF$  shall be  $\sqrt{180} + \sqrt{60} - \sqrt{180} + \sqrt{60}$ .

Lemma.

$B \text{---} M \text{---} G \quad D \text{---} E \text{---} F$   
 $C \text{---} \quad \quad \quad H \text{---}$

If there be the same excess between the first magnitude  $BG$  and the second  $C$  ( $MG$ ), as is between the third magnitude  $DF$  and the fourth  $H$  ( $EF$ ;) then alternately, the same excess shall be between the first magnitude  $BG$  and the third  $DF$ , as is between the second  $C$  and the fourth  $H$ .

For because that (a) to the equals  $BM$ ,  $DE$ , are added the equals  $MG$ ,  $EF$ , that is,  $C$ ,  $H$ ; the excess of the wholes  $BG$ ,  $DF$ , (b) shall be equal to the excess of the parts added  $C$ ,  $H$ . Which was to be demonstrated.

Coroll.

Hence, Four magnitudes Arithmetically proportional, are alternately also Arithmetically proportional.

PROP. LXXX.

To an Apotome or residual line  $AB$  only one rational right-line  $BC$ , being commensurable in power only to the whole  $AB$ , is congruent, or can be joyned.

a hyp 6<sup>o</sup>  
24. 10.  
b 27 10.  
c cor. 7. 2  
p 11 def. 10

a hyp.  
b 15. ax. 1.

a 22. 10.

b 22. 10.

c cor. 7. 2.

d lem. 79. 10.

e hyp. and 27. 10.

f sch. 12. 10.

g 27. 10.

If it be possible let some other line  $BD$  be added to it; (a) then the rectangles  $ACB$ ,  $ADB$ , (b) and so consequently the doubles of them are  $\mu\alpha$ ; wherefore seeing  $ACq + BCq - 2 ACB$  (c)  $= ABq$  (c)  $= ADq + DBq - 2 ADB$ , therefore alternately  $ACq + BCq \div ADq + BDq$  (d)  $= 2 ACB \div 2 ADB$ . But  $ACq + BCq \div ADq + BDq$  (e) it  $\dot{p}$  (f) therefore  $2 ACB \div 2 ADB$   $\dot{p}$ . Which is absurd.

## P R O P. LXXXI.

**A — B — D — C** To a first medial residual-line  $AB$  only one medial right-line  $BC$ , being commensurable only in power to the whole, and comprehending with the whole line a rational-rectangle, can be joyned.

a hyp.

b 16. and 24. 10.

c hyp.

d sch. 12. 10.

e sch. 27. 10.

f 7.2. and lem 79. 10.

g 27. 10.

Conceive  $BD$  to be such a line as may be, joyned to it; then because  $ACq$  and  $BCq$ , as well as  $ADq$  and  $BDq$  (a) are  $\mu\alpha$   $\square$ ; (b) also  $ACq + BCq$ , and  $ADq + BDq$  shall be  $\mu\alpha$ ; (c) but the rectangles  $ACB$ ,  $ADB$ , (d) and so  $2 ACB$  and  $2 ADB$  are  $\dot{p}2$ ; (e) therefore  $2 ACB \div 2 ADB$ , (f) that is,  $ACq + BCq \div ADq + BDq$  is  $\dot{p}$ . (g) Which is absurd.

## P R O P. LXXXII. Plate V. Fig 2.

Upon a second medial-residual-line  $AB$  only one medial right-line  $BC$ , commensurable only in power to the whole, and with it containing a medial-rectangle, can be joyned.

a 4. 2. and 3.

ax. 1.

b hyp.

c 24. 10.

d 23. 10.

e hyp.

f 24. 10.

g lem. 26. 10.

h 1. 6.

k 10. 10.

l 74. 10.

If it be possible, let some other line  $BD$  be added to it; and upon  $EF$   $\dot{p}$  make the rectangle  $EG = ACq + BCq$ ; as also the rectangle  $EL = ADq + BDq$ ; likewise  $EI = ABq$ . Now  $2 ACB + ABq = ACq + BCq = EG$ ; therefore seeing  $EI = ABq$ , (a) also  $KG$  shall be  $= 2 ACB$ , moreover  $ACq$  and  $BCq$  (b) are  $\mu\alpha$   $\square$  (c) therefore  $EG$  ( $ACq + BCq$ ) is  $\mu\alpha$  (d) therefore the breadth  $EH$  is  $\dot{p}$   $\square$   $EF$ . (e) Further, the rectangle  $ACB$  (f) and so  $2 ACB$  ( $KG$ ) is  $\mu\alpha$ ; (d) therefore  $KH$  is also  $\dot{p}$   $\square$   $EF$ . Lastly, because  $ACq + BCq$  ( $EG$ ) (g)  $\square$   $2 ACB$  ( $KG$ ) and  $EG : KG :: EH : KH$ ; (h) therefore  $EH \square HK$ ; (i) therefore  $EK$  is a residual-line, whereto  $KH$  is congruent, by the same reason also shall  $KM$  be congruent to the said  $EK$ . Which is repugnant to the 80. prop. of this Book.

## P R O P.

PROP. LXXXIII.

To a Major-line AB only  $A \text{---} B \text{---} D \text{---} C$   
one right-line BC can be  
joyned being incommensurable in power to the whole, and mak-  
ing together with the whole line that which is composed of their  
squares rational, and the rectangle which is contained under  
them medial.

Conceive any other BD to be congruent to it; There-  
fore whereas  $ACq + BCq$ , and  $ADq + BDq$  (a) are  $\dot{p}z$ ,  
their excess (z (b)  $ACB \div z ADB$ ) (c) is  $\dot{p}y$ . Which is  
absurd; because ACB and ADB are  $\mu a$  by the Hyp.

a 1. hyp.  
b lem. 97.  
10.  
c sch. 27. 10  
d 27. 10.

PROP. LXXXIV.

Unto a line (AB) making  $A \text{---} B \text{---} D \text{---} C$   
with a rational-space a whole  
space medial only one right-line BC can be joined, being in-  
commensurable in power to the whole, and making together with  
the whole that which is composed of their squares medial, and  
the rectangle which is contained under them rational.

Suppose some other BD be congruent also to it; (a)  
then the rectangles ACB, ADB, (b) and so  $z ACB$  and  $z$   
 $ADB$  are  $\dot{p}a$ ; therefore  $z ACB \div z ADB$ , (c) that is,  
 $ACq + BCq \div ADq + BDq$  (d) is  $\dot{p}y$ . Which is absurd;  
since  $ACq + BCq$ , and  $ADq + BDq$  are  $\mu a$  by the  
Hyp.

a hyp.  
b sch. 12.  
10.  
c lem. 79.  
10.  
d sch. 27. 10

PROP. LXXXV. Plate V. Fig. 3.

To a line AB, which with a medial-space makes a whole  
space medial, can be joined only one right-line BC, incom-  
mensurable in power to the whole, and making with the  
whole both that which is composed of their squares me-  
dial, and the rectangle which is contained under them  
medial and incommensurable to that which is composed of  
their squares.

Those things being supposed which are done and  
shewn in the 82. prop. of this Book; it is clear that EH  
and KH are  $\dot{p} \square EF$ . Besides, since,  $ACq + CBq$ ,  
that is, the rectangle EG, (a) is  $\square ACB$ , (b) and so EG  
 $\square z ACB$  (KG;) and  $EG : KG$  (c): : EH : KH, shall EH  
be  $\square KH$ ; therefore EH is a residual-line, and the  
line congruent to it is KH. In like manner may KM  
be shewn to be congruent to, the residual EK, against  
the 8q. prop. of this Book.

a hyp.  
b 14. 10.  
c 1. 6,



## Third Definitions.

**A** Rational-line and a residual being propounded, if the whole be more in power than the line joined to the residual, by the square of a right-line commensurable unto it in length; then

I. If the whole be commensurable in length to the rational-line propounded, it is called a first residual-line.

II. But if the line adjoined be commensurable in length to the rational-line propounded, it is called a second residual-line.

III. If neither the whole nor the line adjoined be commensurable in length to the rational-line propounded, it is called a third residual-line.

Moreover, if the whole be more in power than the line adjoined by the square of a right-line incommensurable to it in length; then

IV. If the whole be commensurable in length to the rational-line propounded, it is called a fourth residual-line.

V. But if the line adjoined be commensurable in length to the rational-line propounded, it is a fifth residual.

VI. If neither the whole nor the line adjoined be commensurable in length to the rational-line propounded, it is termed a sixth residual-line.

PROP. LXXXVI. 87, 88, 89, 90, 91.

A....4 C.....5 B

D—————

E—————F

G

H—————

To find out a first, second, third, fourth, fifth, and sixth residual-line.

Residual-lines are found out by subtracting the less names or parts of binomials from the greater.

Ex. gr. Let  $6 + \sqrt{20}$  be a first binomial, then shall  $6 - \sqrt{20}$  be a first residual. So that it is not necessary to repeat more concerning the finding of them out.

Lemna.

Lemma. Plate V. Fig. 4, 5.

Let AC be a rectangle contained under the right-lines AB, AD. Let AD be drawn forth to E, and DE equally divided in F; and let the rectangle AGE be = FEq, and the rectangles AI, DK, FH, finished. Then let the square LM = AH be made, and the square NO = GI; and the lines NSR, OST, produced.

I say, 1. The rectangle AI = LM + NO = TOq + SOq, which appears by the constr.

2. The rectangle DK = LO. For because the rectangle AGE (a) = FEq (b) thence are AG, FE, GE :: (c) and so AH, FI, GI :: (a) that is, LM, FI, NO ::; but LM, LO, NO (d) are ::; therefore FI = (e) LO (f) = DK = (g) NM.

3. Hence, AC = AI - DK - FI = LM + NO - LO - NM = TR.

4. It is manifest that DF, FE, DE, are  $\square$ .

5. If AE  $\square$  DE, and AE  $\square \sqrt{AEq - DEq}$ , (k) then shall AG, GE, AE be  $\square$ .

6. Also, because AE (l)  $\square$  DE, (m) thence shall AE, FE, be  $\square$ ; (n) and so AI, FI, that is, LM + NO and LO are  $\square$ .

7. Because AG\*  $\square$  GE, (n) shall AH, GI, that is, LM, NO be  $\square$ .

8. But because AE (l)  $\square$  DE, (o) therefore shall FE, GE be  $\square$ , (n) and so the rectangle FI  $\square$  GI, that is, LO  $\square$  NO; wherefore seeing LO : NO (p) :: TS : SO; (q) therefore shall TS, SO be  $\square$ .

9. If AE be put  $\square \sqrt{AEq - DEq}$ , then shall AG, GE, AE be  $\square$ .

10. (f) Wherefore the rectangles AH, GI, that is, TOq, SOq shall be  $\square$ .

a constr.  
b 17. 6.  
c 1. 6.  
d scb. 22. 6.  
e 9 5.  
f 36. 1.  
g 43. 1.  
h 16. 10.  
k 18. and  
10. 10.  
l hyp.  
m 13. 10.  
n 1. 6. and  
10. 10.  
\* before.  
o 14. 10.  
p 2. 6.  
q 10. 10.  
r 19. 10. &  
17. 10.  
s 1. 6. and  
10. 10.

PROP. XCII. Fig. 4, 5.

If a space AC be contained under a rational-line AB, and a first residual-line AD (AE - DE) the right-line TS, which containeth the space AC in power, is a residual-line.

M 4

Use

a hyp.  
b 12. 10.  
c 20. 10.  
d lem. 91.  
10.  
e 74. 10.

Use the foregoing *Lemma* for a preparation to the demonstration of this prop. Therefore  $TS = \sqrt{AC}$ . Also AG, GE, AE, are  $\perp$ ; therefore since AE  $\perp$  (a) AB  $\rho$ , (b) also AG and GE shall be  $\perp$  AB, (c) therefore the rectangles AH and GI, that is, TOq and SOq are  $\rho\alpha$ . (d) Likewise TO, SO, are  $\rho \perp$  (e) and consequently TS is a residual-line. Which was to be demonstrated.

## P R O P. XCIII. Plate V. Fig. 4, 5.

If a space AC be contained under a rational line AB, and a second residual AD (AE—DE) the right-line TS, containing the space AC in power, is a first medial residual-line.

a hyp.  
b 13. 10.  
c 22. 10.  
d lem. 74.  
10.  
e hyp.  
f 20. 10.  
g 75. 10.

Again, by the foregoing *Lemma*, AG, GE, AE are  $\perp$ ; therefore (a) since AE is  $\rho \perp$  AB, (b) also AG, GE, shall be  $\rho \perp$  AB, (c) therefore the rectangles AH, GI, that is, TOq, SOq are  $\mu\alpha$ ; (d) likewise TO  $\perp$  SO. Lastly, because DE (e)  $\perp$  AB  $\rho$ , (f) the right-angle DI, and the half thereof DK or LO, that is, TOS shall be  $\rho\nu$ ; (g) from whence it follows that TS ( $\sqrt{AC}$ ) is a first medial residual. Which was to be demonstrated.

## P R O P. XCIV. Fig. 4, 5.

If a space AC be contained under a rational-line AB and a third residual AD (AE—DE) the right-line TS containing in power the space AC is a second medial residual-line.

a hyp.  
b 22. 10.  
c 24. 10.  
d 76. 10.

As in the former, TO and SO are  $\mu$ . Therefore because DE (a) is  $\rho \perp$  AB, (b) the rectangle DI, (c) and so DK, or TOS, shall be  $\mu\nu$ ; therefore  $TS = \sqrt{AC}$  is a second medial residual. Which was to be demonstrated.

## P R O P. XCV. Fig. 4, 5.

If a space AC be contained under a rational-line AB and a fourth residual AD (AE—DE) the right-line TS containing the space AC in power, is a Minor-line.

a lem. 91. 10.  
b hyp.  
c 20. 10.  
d 77. 10.

As before, TO (a)  $\perp$  SO. Therefore because AE (b) is  $\rho \perp$  AB, (c) shall AI (TOq+SOq) be  $\rho\nu$ , but, as before, the rectangle TOS is  $\mu\nu$ ; (d) therefore  $TS = \sqrt{AC}$  is a Minor-line. Which was to be demonstrated.



PROP. XCVI. Plate V. Fig. 4. 5.

If a space AC be contained under a rational-line AB and a fifth residual AD (AE - DE) the right-line TS containing in power the space AC, is a line which maketh with a rational space the whole space medial.

For again TO  $\sqsupset$  SO. Therefore since AE (a) is  $\dot{p}$   $\sqsupset$  AB (b) also AI, that is, TOq + SOq shall be  $\mu\nu$ . But, as in 93. the rectangle TOS is  $\dot{p}\nu$ ; (c) whence TS =  $\sqrt{AC}$  is a line which with  $\dot{p}\nu$  makes a whole  $\mu\nu$ . Which was to be demonstrated.

a hyp.  
b 22. 10.  
c 78. 10.

PROP. XCVII. Fig. 4. 5.

If a space AB contained under a rational-line AB, and a fifth residual AD (AE - DE) the right-line TS containing in power the space AC is a line making with a medial rectangle, a whole space medial.

As often above TO  $\sqsupset$  SO. Also, as in 96. TOq + SOq is  $\mu\nu$ , but the rectangle TOS is  $\dot{p}\nu$ , as in 94. (a) Lastly, TOq + SOq  $\sqsupset$  TOS (b) therefore, TS =  $\sqrt{AC}$  is a line which with  $\mu\nu$  makes a whole  $\mu\nu$ . Which was to be dem.

a lem. 91.  
10.  
b 79. 10.

Lemma. Fig. 6.

Upon a right-line DE \* apply the rectangles DF = ABq, and DH = ACq, and IK = BCq, and let GL be bisected in M, and the line MN drawn parallel to GF.

\* cor. 16. 6.

Then 1. The rectangle DK is = ACq + BCq, as the construction manifests.

2. The rectangle ACB = GN or MK. For DK (a) = ACq + BCq (b) = 2 ACB + ABq; but ABq (a) = DF. therefore GK (c) = 2 ACB; and consequently GN or MK = ACB.

a constr.  
b 7. 2.  
c 3. ax. 1.  
d 7. ax. 1.  
e 1. 6.  
f 17. 6.

3. The rectangle DIL = MLq. For because ACq : ACB (e) :: ACB : BCq, that is, DH : MK :: MK : IK. (e) thence is DI : ML :: ML : IL (f) therefore DIL = MLq.

4. If AC be taken  $\sqsupset$  BC, then DK shall be  $\sqsupset$  ACq. For ACq + BCq (DK) (g)  $\sqsupset$  ACq.

g 16. 10.

5. Likewise DL  $\sqsupset$   $\sqrt{DLq - GLq}$ . For because DH (ACq)  $\sqsupset$  IK (BCq) (b) thence shall DI be  $\sqsupset$  IL; (k) therefore  $\sqrt{DLq - GLq} \sqsupset$  DL.

h 10. 10.  
k 18. 10.  
lem. 26. 10.  
m 10. 10.

6. Also DL  $\sqsupset$  GL. For ACq + BCq  $\sqsupset$  (l) 2 ACB. That is, DK  $\sqsupset$  GK; (m) therefore DL  $\sqsupset$  GL.

7. But

7. But if AC be taken  $\sqsupset$  BC, then DL shall be  $\sqsupset \sqrt{DLq - GLq}$ .

PROP. XCVIII. Plate V. Fig. 6.

a hyp.  
b lem. 97.  
10.  
c sch. 12.  
10.  
d 21. 10.  
e 22. and  
24. 10.  
f 23. 10.  
g 13. 10.  
h sch. 12.  
10.  
k 74. 10.  
l def. 85. 10.  
m lem. 97.  
10.

The square of a residual-line AB ( $AC - BC$ ) applied to a rational-line DE, makes the breadth DG a first residual-line. Do as is enjoined in the Lemma next preceding. Then because AC, BC, (a) are  $\rho \sqsupset$ ; (b) also DK ( $ACq + BCq$ ) shall be  $\sqsupset ACq$ . (c) Therefore DK is  $\rho \sqsupset$ ; (d) wherefore DL is  $\rho \sqsupset DE$ . (e) Likewise the rectangle GK (2 ACB) is  $\mu\nu$ ; (f) therefore GL is  $\rho \sqsupset DE$ , (g) and consequently DL  $\sqsupset GL$ . (h) But  $DLq \sqsupset GLq$ , (i) therefore DG is a residual, (l) and that of the first order (because (m)  $AC \sqsupset BC$ , and therefore DL  $\sqsupset \sqrt{DLq - GLq}$ .) Which was to be demonstrated.

PROP. XCIX. Fig. 6.

a hyp.  
b lem. 97.  
10.  
c 24. 10.  
d 23. 10.  
e hyp. and  
sch. 12. 10.  
f 21. 10.  
g 13. 10.  
h sch. 12.  
10.  
k 74. 10.  
l lem. 97.  
10.  
m 2. def.  
85. 10.

The square of a first medial residual-line AB ( $AC - BC$ ) applied to a rational-line DE, makes the breadth DG a second residual-line.

Supposing the foregoing Lemma; because AC and BC (a) are  $\mu \sqsupset$ , (b) thence shall DK ( $ACq + BCq$ ) be  $\sqsupset ACq$ ; (c) wherefore DK is  $\mu\nu$ ; (d) therefore DL is  $\rho \sqsupset DE$ ; (e) also GK (2 ACB) is  $\rho$ ; (f) therefore GL is  $\rho \sqsupset DE$ ; (g) wherefore DL  $\sqsupset GL$ . (h) But  $DLq \sqsupset GLq$ ; (i) therefore DG is a residual-line. And because DL is  $\sqsupset \sqrt{DLq - GLq}$ , (m) therefore shall DG be a second residual. Which was to be demonstrated.

PROP. C. Fig. 6.

a 23. 10.  
b lem. 26.  
10.  
c 1. 6. and  
10. 10.  
d sch. 12. 10.  
e 74. 10.  
f 3. def. 85.  
10.  
g lem. 97.  
10.

The square of a second medial residual-line AB ( $AC - BC$ ) applied to a rational-line DE, makes the breadth DG a third residual-line.

Again DK is  $\mu\nu$ . (a) wherefore DL is  $\rho \sqsupset DE$ ; also GK is  $\mu\nu$ , (a) whence GL is  $\rho \sqsupset DE$ ; (b) likewise DK  $\sqsupset GK$ ; (c) wherefore DL  $\sqsupset GL$ ; (d) but  $DLq \sqsupset GLq$ . (e) therefore DG is a residual-line, and that of (f) the third order, (g) because DL  $\sqsupset \sqrt{DLq - GLq}$ . Which was to be demonstrated.

PROP.

PROP. CI. Plate V. Fig. 6.

The square of a Minor-line AB (AC—BC) applied to a rational-line DE, makes the breadth DG a fourth residual.

As before, ACq+BCq, that is DK, is  $\rho\nu$ . (a) therefore DL is  $\rho \sqsubset$  DE. but the rectangle ACB, and so GK (2 ACB) \* is  $\mu\nu$ . (b) wherefore GL is  $\rho \sqsubset$  DE. (c) therefore DL  $\sqsubset$  GL. (d) but DLq  $\sqsubset$  GLq. and because \* ACq  $\sqsubset$  BCq, (e) thence shall DL be  $\sqsubset \sqrt{\text{DLq} - \text{GLq}}$ . (f) therefore DG has the conditions required to a fourth residual. Which was to be demonstrated.

PROP. CII. Fig. 6.

The square of a line AB (AC—BC) which makes with a rational space the whole space medial, applied to a rational-line DE, makes the breadth DG a fifth residual-line.

For, as above, DK is  $\mu\nu$ . (a) wherefore DL is  $\rho \sqsubset$  DE also GK is  $\rho\nu$ . (b) whence GL is  $\rho \sqsubset$  DE. (c) therefore DL  $\sqsubset$  GL. (d) but DLq  $\sqsubset$  GLq. Moreover DL (e)  $\sqsubset \sqrt{\text{DLq} - \text{GLq}}$ . wherefore DG (f) is a fifth residual. Which was to be demonstrated.

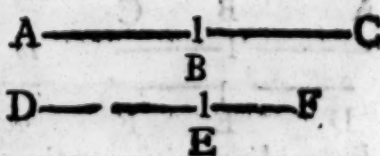
PROP. CIII. Fig. 6.

The square of a line AB (AC—BC) making with a medial space the whole space medial, applied to a rational-line DE, makes the breadth DG a sixth residual-line.

As above, DK and GK are  $\mu a$ ; (a) wherefore DL and GL are  $\rho \sqsubset$  DE. also DK (b)  $\sqsubset$  GK. (c) whence DL  $\sqsubset$  GL. (d) therefore DG is a residual. (b) And whereas ACq  $\sqsubset$  BCq. and so DL  $\sqsubset \sqrt{\text{DLq} - \text{GLq}}$ , (e) therefore DG shall be a sixth residual. Which was to be demonstrated.

PROP. CIV.

A right-line DE commensurable in length to a residual AB (AC—BC) is itself also a residual, and of the same order.



Lemma.

Let AB : DE :: AC : DF, and AB  $\sqsubset$  DE;

a 21. 10.  
\* hyp.  
b 23. 10.  
c 13. 10.  
d sch. 12.  
10.  
e lem. 97.  
10.  
f 4. def. 85.  
10.

a 23. 10.  
b 21. 10.  
c 13. 10.  
d sch. 12.  
10.  
e lem. 97.  
10.  
f 5. def. 85.  
10.

a 23. 10.  
b hyp. and  
lem. 97. 10.  
c 10. 10.  
d 74. 10.  
e 6. def. 85.  
10.



a lem. 66.

10.

b 10. 10.

a 12. 6.

b lem. 103.

10.

c hyp.

d 67. 10.

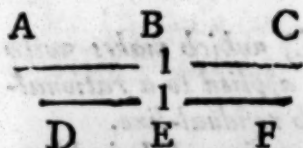
e by def. 85.

10.

I say  $AC + BC \sqsubset DF + EF$ . For  $AC : BC$  (a) :  $DF : EF$ . therefore by compounding  $AC + BC : BC$  :  $DF + EF : EF$  therefore by permutation  $AC + BC : DF + EF :: BC : EF$ . (a) but  $BC \sqsubset EF$ ; (b) therefore  $AC + BC \sqsubset DF + EF$ . Which was to be dem.

(a) Make  $AB : DE :: AC : DF$ , (b) therefore  $AC + BC \sqsubset DF + EF$ . therefore seeing  $AC + BC$  (c) is a binomial, (d)  $DF + EF$  shall be a binomial too, and of the same order. (e) wherefore  $DF - EF$  is a residual of the same order with  $AC - BC$ . Which was to be demonstrated.

## P R O P. CV.



A right-line DE commensurable to a medial residual-line AB ( $AC - BC$ ) is it self a medial residual and of the same order.

Again (a) make  $AB : DE :: AC : DF$ ; (b) whence  $AC + BC \sqsubset DF + EF$ . (c) therefore  $DF + EF$  is a binomial of the same order with  $AC + BC$ , (d) and consequently  $DF - EF$  shall be a medial residual of the same order with  $AC - BC$ . Which was to be dem.

a 12. 6.

b lem. 103.

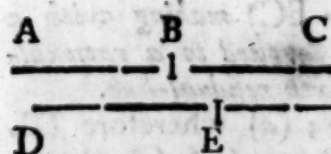
10.

c 68. 10.

d 75. and

76. 10.

## P R O P. CVI.



A right-line DE commensurable to a Minor-line AB ( $AC - BC$ ) is it self also a Minor-line.

Make  $AB : DE :: AC : DF$ . (a) then is  $AC + BC \sqsubset DF + EF$ . but  $AC + BC$  (b) is a Major line; (c) therefore  $DF + EF$  is also a Major line; (d) and consequently  $DF - EF$  is a Minor line. Which was to be dem.

a lem. 103.

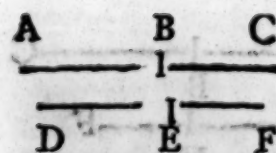
10.

b hyp.

c 69. 10.

d 77. 10.

## P R O P. CVII.



A right-line DE commensurable to a line AB ( $AC - BC$ ) which makes with a rational space the whole space medial, is it self also a line making with a rational space the whole space medial.

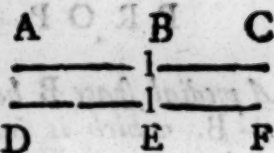
For, accordingly as in the former, we may shew  $DF + EF$  to contain in power  $\rho\nu$  and  $\mu\nu$ . (a) whence  $DF - EF$  is a line making, &c.

a 78. 10.

## P R O P.

PROP. CVIII.

A right-line DE commensurable  
with a line AB (AC - BC) which  
with a medial space makes the  
whole space medial, is it self a line  
making with a medial space the  
whole space medial.



For according to the preceding  $DF + EF$  shall con-  
tain in power  $2 \mu a$ . (a) therefore  $DF - EF$  shall be, as

a 79. 10.

PROP. CIX. Plate V. Fig. 7.

A medial rectangle B being taken from a rational rectan-  
gle A+B, the right-line H which containeth in power the  
space remaining A, is one of these two irrational-lines, viz.  
either a residual-line, or a Minor-line.

Upon CD  $\rho$  make the rectangles  $CI = A + B$ , and  
 $FI = B$ . whence  $CE$  (a)  $= A = Hq$ . wherefore because  $CI$   
(b) is  $\rho v$ . (c) therefore  $CK$  is  $\rho \sqsubset CD$ ; but because  $FI$  (b)  
(d) shall  $FK$  be  $\rho \sqsubset CD$ ; (e) whence  $CK \sqsubset FK$ ;  
(f) therefore  $CF$  is a residual-line. Wherefore if  $CK$   
(g) be  $\sqrt{CKq - FKq}$ . (g) then  $CF$  shall be a first  
residual; (b) therefore  $\sqrt{CE}$  (H) is a residual-line. But  
(k) then  $CF$  shall be a fifth  
residual; and consequently  $H$  ( $\sqrt{CE}$ ) (l) shall be a  
Minor-line. Which was to be demonstrated.

a 3. ax. 1.  
b hyp. and  
constr.

c 21. 10.  
d 23. 10.  
e 13. 10.  
f 74. 10.  
g 1. def. 85.  
10.  
h 92. 10.  
k 4. def. 85.  
10.  
l 95. 10.

PROP. CX. Fig. 7.

A rational rectangle B being taken away from a medial  
rectangle A+B, other two irrational-lines are made, name-  
ly; either a first medial residual-line, or a line making with  
a rational space the whole space medial.

Upon CD the propounded  $\rho$  make the rectangle  $CI$   
 $= A + B$ , and  $FI = B$  (a) whence  $CE = A = Hq$ .  
Therefore because  $CI$  (b) is  $\mu v$ ; (c) shall  $CK$  be  $\rho \sqsubset CD$ .  
but because  $FI$  (b) is  $\rho v$  (d) thence  $FK$   $\rho \sqsubset CD$ . (e)  
whence  $CK \sqsubset FK$ . (f) therefore  $CF$  is a residual, (g)  
and that a second. If  $CK \sqsubset \sqrt{CKq - FKq}$ , (b) then  
 $H$  ( $\sqrt{CE}$ ) is a first medial residual. But if  $CK \sqsubset \sqrt{CKq - FKq}$ , (k) then shall  $CF$  be a fifth residual; and  
(l) consequently  $H$  ( $\sqrt{CE}$ ) shall be a line making  $\mu v$   
with  $\rho v$ . Which was to be demonstrated.

a 3. ax. 1.  
b hyp. and  
constr.

c 23. 10.  
d 21. 10.  
e 13. 10.  
f 74. 10.  
g 2. def. 85.  
10.  
h 93. 10.  
k 5. def. 85.  
10.  
l 96. 10.

PROP.

## P R O P. CXI. Plate V. Fig. 7.

*A medial space B being taken away from a medial space A + B, which is incommensurable to the whole A + B, the other two irrational-lines are made, viz. either a second medial residual-line, or a line making with a medial space the whole space medial.*

- a 3. ax. 1.  
b 23. 10.  
c hyp.  
d 10. 10.  
e 74. 10.  
f 3. def. 85.  
10.  
g 94. 10.  
h 6. def. 85.  
10.  
k 97. 10.

Upon CD  $\rho$  make the rectangles CI = A + B, and FI = B. (a) wherefore CE = A = Hq. Because therefore CI is  $\mu\nu$ , (b) thence CK is  $\rho \sqsubset$  CD. and in like manner FK  $\rho \sqsubset$  CD. Likewise because CI (c)  $\sqsubset$  FI (d) therefore CK  $\sqsubset$  FK (e) wherefore CF is a residual (f) namely a third, If CK  $\sqsubset \sqrt{CKq - FKq}$ , (g) whence H ( $\sqrt{CE}$ ) shall be a second medial residual; but CK  $\sqsubset \sqrt{CKq - FKq}$ . (h) then shall CF be a first residual; (i) wherefore A shall be a line making  $\mu\nu$  with  $\mu$ . Which was to be demonstrated.

## P R O P. CXII. Fig. 8.

*A residual-line A is not the same with a binomial-line.*

- a 98. 10.  
b 74. 10.  
c 1. def. 85.  
10.  
d 37. 10.  
e 1. def. 48.  
10.  
f 12. 10.  
g cor. 16.  
10.  
h sch. 12.  
10.  
k 14. 10.  
l 74. 10.

Upon BC propounded  $\rho$  make the rectangle CD = Aq. Therefore seeing A is a residual, (a) BD shall be a first residual, to which let DE be the line congruent or that may be adjoined; (b) wherefore BE, DE, are  $\rho \sqsubset$ ; (c) and BE  $\sqsubset$  BC. If you conceive A to be a binomial then BD is a first binomial, whose names let be BF, FD and let BF be  $\sqsubset$  FD; (d) therefore BF, FD are  $\rho \sqsubset$  and BF (e)  $\sqsubset$  BC. therefore since BC  $\sqsubset$  BE, (f) shall BE be  $\sqsubset$  BF. (g) and thence BE  $\sqsubset$  FE, therefore FE is  $\rho$ . Likewise because BE  $\sqsubset$  DE, (h) shall FE be  $\sqsubset$  DE; (i) wherefore FD is a residual, and FD is  $\rho$ . but it was shewn  $\rho$ . which are repugnant therefore A is falsely conceived to be a binomial. Which was to be demonstrated.



The names of the 13 irrational-lines differing one from another.

1. A Medial-line.
2. A binomial-line; of which there are six species.
3. A first bimedial-line.
4. A second bimedial.
5. A Major-line.
6. A line containing in power a rational superficies, and a medial superficies.
7. A line containing in power two medial superficies.
8. A residual-line; of which there are also six kinds.
9. A first medial residual-line.
10. A second medial residual-line.
11. A Minor-line.
12. A line making with a rational superficies the whole superficies medial.
13. A line making with a medial superficies the whole superficies medial.

Since the differences of breadths argue differences of right-lines, whose squares are applied to some rational-line, and it is demonstrated in the preced. Propositions that the breadths which arise from applying of the squares of these lines to differ one from another, it evidently follows that these 13 lines do also differ one from another.

PROP. CXIII. Plate V. Fig. 8.

The square of a rational-line A applied to a binomial C (BD + DC) makes the breadth EC a residual-line, whose names EH, CH, are commensurable to the names BD, DC, of the binomial-line, and in the same proportion (EH : BD :: CH : DC;) and moreover, the residual-line EC which is made, is of the same order with BC the binomial.

Upon DC the less name (a) make the rectangle DF = Aq = BE, whence BC : CD (b) :: FC : CE. therefore by division, BD : DC :: FE : EC. And whereas D (c)  $\sqsubset$  DC, (d) thence FE shall be  $\sqsubset$  EC; Take EG = EC, and make FG : GE :: EC : CH. Then EH, and CH shall be the names of the residual EC, whereunto all is agreeable that is propounded in the theorem. or by compounding, FE : GE (EC) :: EH : CH. therefore FH : EH (e) :: EH : CH (f) :: FE : EC (f) :: BD : DC. wherefore since BD (g)  $\sqsupset$  DC (b) thence shall EH be

a' cor. 16.6.

b 14. 6.

c hyp.

d 14. 5.

e 12. 5.

f before.

g hyp.

h 10. 10.

k cor. 20. 6. be  $\square$  EH, (b) and FHq  $\square$  EHq. Therefore because  
 l 16. 10. FHq : EHq (k) FH : CH : b) shall FH be  $\square$  CH; (l)  
 m 21. 10. and so FC  $\square$  CH. Moreover CD (g) is  $\dot{p}$ , and DF (Aq)  
 n scb. 12. (g) is  $\dot{p}$ ; (m) therefore FC is  $\dot{p}$ ,  $\square$  CD, whence also CH  
 10. is  $\dot{p}$   $\square$  CD; (n) therefore EH, CH are  $\dot{p}$  and  $\square$ , as be-  
 o 74. 10. fore; (o) therefore EC is a residual-line, to which CH may  
 p 10. 10. be joined. Furthermore EH : CH (f) : : BD : DC, and  
 q 15. 10. so by permutation EH : BD : : CH : DC; whence be-  
 r 12. 10. cause CH (f)  $\square$  DC, (p) shall EH be  $\square$  BD. But  
 f 1. def. 48. suppose BD  $\square$   $\sqrt{BDq - DCq}$ , (q) then shall EH be  
 10.  $\square$   $\sqrt{EHq - CHq}$ . Also if BD  $\square$   $\dot{p}$  propounded,  
 t 1. def. 85. then shall EH be  $\square$  to the same  $\dot{p}$ . (f) that is, if BC be  
 10. a first binomial, (r) EC shall be a first residual. In like  
 u 2. def. 48. manner, if DC be to the  $\square$  propounded  $\dot{p}$ , (r) then is  
 10. CH  $\square$  to the same  $\dot{p}$ . (u) that is, if BC be a second  
 x 2. def. 85. binomial, (x) EC shall be a second residual. And if  
 10. this be a third binomial, then that shall be a third re-  
 y 15. 10. sidual, &c. But if BD be  $\square$   $\sqrt{BDq - DCq}$ , (y) then  
 shall EH be  $\square$   $\sqrt{EHq - CHq}$ . therefore if BC be a  
 4, 5, or 6 binomial, EG shall be likewise a 4, 5, or 6,  
 residual. Which was to be demonstrated.

P R O P. CXIV. Plate V. Fig. 9.

The square of a rational line A applied to a residual-line  
 BC (BD - CD) makes the breadth BE a binomial; whose  
 names BE, GE are commensurable to the names BD, BC of  
 the residual-line BC, and in the same proportion; and more-  
 over, the binomial-line which is made (BE) is of the same  
 order with the residual-line (BC.)

a cor. 16. 6. (a) Make the rectangle DF = Aq. and BF : FE (b)  
 b 12. 6. : : EG : GF; whence for that DF = Aq = CE, (c) there-  
 c 14. 6. fore BD : BC : : BE : BF. therefore by conversion of pro-  
 d 19. 5. portion BD : CD : : BE : FE : : EG : GF : : (d) GB : EG,  
 e hyp. but BD (e)  $\square$  CD; (f) therefore BG  $\square$  GE. therefore  
 f 10. 10. because BGq : GEq (g) : : BG : GF. (b) shall BG be  $\square$  GF  
 g cor. 20. 6. (k) and so BG  $\square$  BF. moreover BD (e) is  $\dot{p}$ , and the rec-  
 h 10. 10. tangle DF (Aq) (e) is  $\dot{p}$ . (l) therefore BF is  $\dot{p}$   $\square$  BD; (m)  
 k cor. 16. 10. therefore also BG is  $\dot{p}$   $\square$  BD. (n) therefore BG, GE  
 l 21. 20. are  $\dot{p}$   $\square$ . (a) wherefore BE is a binomial. Lastly, be-  
 m 12. 10. cause BD : CD : : BG : GE. and by permutation BD : BG  
 n scb. 12. 10. : : CD : GE, and BD  $\square$  BG (p) thence shall CD be  $\square$   
 o 37. 10. GE. therefore if CB be a first residual, BE shall be a first  
 p 10. 10. binomial, &c. as in the prec. therefore, &c.

P R O P

PROP. CXV. Plate V. Fig. 10.

If a space AB be contained under a residual-line AC (CE — AE) and a binomial CB, whose names CD, DB, are commensurable to the names CE, AE, of the residual-line, and in the same proportion (CE : AE :: CD : DB.) then the right-line F which containeth in power that space AB, is rational.

Let G be  $\dot{\rho}$ . and make the rectangle CH = Gq; (a) then shall BH (HI — IB) be a residual-line, and HI (a)  $\perp$  CD (b)  $\perp$  CE. (a) and BI  $\perp$  DB. (a) and HI : BI :: CD : DB (b) :: CE : EA. therefore by permutation HI : CE :: BI : EA. (c) therefore BH : AC :: HI : CE :: BI : EA. wherefore since (d) HI  $\perp$  CE, (e) thence BH  $\perp$  AC. (f) therefore the rectangle HC  $\perp$  BA. But HC (Gq) (b) is  $\dot{\rho}\nu$ . (g) therefor BA (Fq) is  $\dot{\rho}\nu$ . and consequently F is  $\dot{\rho}$ . Which was to be demonstrated.

a 113. 10.  
b hyp.  
c 19. 5.  
d 12. 10.  
e 10. 10.  
f 1. 6. and  
10. 10.  
g sch. 12.  
10.

Coroll.

Hereby it appears that a rational superficies may be contained under two irrational right-lines.

PROP. CXVI. Fig. 11.

Of a medial-line AB are produced infinite irrational-lines BE, EF, &c. whereof none is of the same kind with any of the precedent.

Let AC be propounded  $\dot{\rho}$ . and AD a rectangle contained under AC, AB; (a) therefore AD is  $\dot{\rho}\nu$ . Take BE =  $\sqrt{AD}$ . (b) then BE is  $\dot{\rho}$ , and the same with none of the former. For no square of any of the former being applied to  $\dot{\rho}$ , makes the breadth medial. Let the rectangle DE be finished, (a) then DE shall be  $\dot{\rho}\nu$ , and (b) consequently EF ( $\sqrt{DE}$ ) shall be  $\dot{\rho}$ , and not the same with any of the former, for no square of the former being applied to  $\dot{\rho}$ , makes the latitude BE; therefore, &c.

a lem. 38.  
10.  
b 11. 10.]



## P R O P. CXVII. Plate V. Fig. 12.

Let it be required to shew that in square figures BD, the diameter AC is incommensurable in length to the side AB.

a 47. 1.

b cor. 24. 8.

c 9. 10.

For  $ACq : ABq$  (a) :: 2 : 1 (b) :: not Q : Q. (c) therefore AC  $\nmid$  AB. Which was to be demonstrated

This Theorem was of great note with the ancient Philosophers ; so that he who did not understand it was esteemed by Plato undeserving the name of a man, but rather to be reckoned among brutes.

The End of the Tenth Book.



T H

## The ELEVENTH BOOK

O F

## EUCLID'S

## ELEMENTS.

*Definitions.*

**I.** Solid is that which hath length, breadth and thickness.

**II.** The term, or extreme of a solid is a Superficies.

**III.** A right-line AB, Plate V. *Fig. 13.* is perpendicular to a Plane CD, when it makes right-angles ABD, ABE, ABF, with all the right-lines BD, BE, BF, that touch it, and are drawn in the said Plane.

**IV.** A Plane AB, *Fig. 14.* is perpendicular to a Plane CD, when the right-lines FG, HK, drawn in one Plane AB to the line of common section of the two Planes EB, and making right-angles therewith, do also make right-angles with the other Plane CD.

**V.** The inclination of a right-line AB, *Fig. 15.* to a Plane CD, is when a perpendicular AE is drawn from A the highest point of that line AB to the plane CD, and another line EB drawn from the point E, which the perpendicular AE makes in the plane CD, to the end B of the said line AB which is in the same plane, whereby the angle ABE which is contained under the infisting-line AB, and the line drawn in the plane EB is acute.

**VI.** The inclination of a plane AB, *Fig. 16.* to a plane CD, is an acute angle FGH contained under the right-lines FH, GH which being drawn in either of the planes AB, CD to the same point H of the common section BE, make right-angles FHB, GHB, with the common section BE.

VII. Planes are said to be inclined to other planes in the same manner, when the said angles of inclination are equal one to another.

VIII. Parallel planes are those which being prolonged never meet.

IX. Like solid figures are such as are contained under like planes equal in number.

X. Equal and like solid figures are such as are contained under like planes equal both in multitude and magnitude.

XI. A solid angle is the inclination of more than two right-lines which touch one another, and are not in the same superficies.

*Or thus ;*

A solid angle is that which is contained under more than two plane angles not being in the same superficies but consisting all at one point.

XII. A Pyramid is a solid figure comprehended under divers planes set upon one plane (which is the base of the pyramid,) and gathered together to one point.

XIII. A Prism is a solid figure contained under planes, whereof the two opposite are equal, like, and parallel ; but the others are parallelograms.

XIV. A Sphere is a solid figure made when the diameter of a semicircle abiding unmoved, the semicircle is turned round about, till it return to the same place from whence it began to be moved.

*Coroll.*

Hence, all the rays drawn from the center to the superficies of a sphere, are equal amongst themselves.

XV. The Axis of a sphere, is that fixed right-line about which the semicircle is moved.

XVI. The Center of a sphere, is the same point with the center of the semicircle.

XVII. The Diameter of a sphere, is a right-line drawn thro' the center, and terminated on either side in the superficies of the sphere.

XVIII. A Cone is a figure made, when one side of a rectangled triangle (*viz.* one of those that contain the right angle) remaining fixed, the triangle is turned round about till it return to the place from whence



it first moved. And if the fixed right-line be equal to the other which containeth the right-angle, then the Cone is a rectangled Cone: But if it be less, it is an obtuse-angled Cone; if greater, an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the triangle is moved.

XX. The Base of a Cone is the circle, which is described by the right-line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the sides thereof, (namely, which contain the right-angle) abiding fix'd, till the parallelogram be turned about to the same place, where it began to move.

XXII. The Axis of a Cylinder is that quiescent right-line, about which the parallelogram is turned.

XXIII. And the Bases of a Cylinder are the circles which are described by the two opposite sides in their motion.

XXIV. Like Cones and Cylinders, are those both whose Axes and Diameters of their Bases are proportional.

XXV. A Cube is a solid figure contained under six equal squares.

XXVI. A Tetraedron is a solid figure contained under four equal and equilateral triangles.

XXVII. An Octaedron is a solid figure contained under eight equal and equilateral triangles.

XXVIII. A Dodecaedron is a solid figure contained under twelve equal, equilateral, and equiangular Pentagones.

XXIX. An Icosaedron is a solid figure contained under twenty equal and equilateral triangles.

XXX. A Parallelepipedon is a solid figure contained under six quadrilateral figures, whereof those which are opposite are parallel.

XXXI. A solid figure is said to be inscribed in a solid figure, when all the angles of the figure inscribed are comprehended either within the angles, or in the sides, or in the planes of the figure wherein it is inscribed.

XXXII. Likewise a solid figure is then said to be circumscribed about a solid figure, when either the angles, or sides, or planes of the circumscribed figure touch all the angles of the figure which it contains.

## PROPOSITION I. Plate V. Fig. 17.

One part AC of a right-line cannot be in a plane superficies, and another part of it CB above the same.

a 10. ax. 1.

Produce AC in the plane directly to F. If you conceive CB to be drawn strait from AC, then two right-lines AB, AF, have one common segment AC. (a) Which is impossible.

## PROP. II. Fig. 18.

If two right-lines AB, CD, cut one another, they are in the same plane: And every triangle DEB is in one and the same plane.

a 10. ax. 1.

For imagine EFG, part of the triangle DEB, to be in one plane, and the part FDGB to be in another, then EF part of the right-line ED is in a plane, and the other part elevated upwards. (a) Which is absurd. Therefore the triangle EDB is in one and the same plane; and so also are the right-lines ED, EB; (a) wherefore the whole lines AB, DC, are in one plane. Which was to be demonstrated.

## PROP. III. Fig. 19.

If two planes AB, CD, cut one the other, their common section EF is a right-line.

a 1. post. 1.

b 14. ax. 1.

If EF the common section be not a right-line, (a) then in the plane AB draw the right-line EGF; (a) and in the plane CD the right-line EHF. therefore two right-lines EGF, EHF include a superficies. (b) Which is absurd.

## PROP. IV. Fig. 20.

If at E the common section of two right-lines AB, CD, a right-line EF stands at right-angles to them, it shall also be at right-angles to the plane ACBD drawn thro' the said lines.

a constr.

b 15. 1.

c 1. 1.

d f b. 34. 1.

e 29. 1.

Take EA, FC, EB, ED, equal one to the other, and join the right-lines AC, CB, BD, AD draw any right-line GH thro' E, and join FA, FC, FD, FB, FG, FH. Because AE is (a) = EB, and DE (a) = EC, and the angle AED (b) = CEB (c) therefore AD is = CB, (d) likewise AC = DB; (d) therefore AD is parallel to CB (d) and AC to BD. (e) wherefore the angle GAE = EBH

EBH, and the angle AGE = EHB. But also AE (f) = EB; (g) therefore GE = EH, (g) and AG = BH. whence by reason of the right-angles, by the hyp. and so equal, at E, (b) the bases FA, FC, FB, FD, are equal. Therefore the triangles ADF, FBC, are equilateral one to another, (k) and thence the angle DAF = BCF. Therefore in the triangles AGF, FBH, the sides FG, FH (l) are equal; and so by consequence the triangles FEG and FEH are mutually equilateral; (m) therefore the angles FEG, FEH are equal, and (n) so right-angles. In like manner, FE makes right-angles with all the lines drawn thro' E in the plane AD, BC, (o) and is therefore perpendicular to the said plane.

f constr.

g 26. 1.

h 4. 1.

k 8. 1.

l 4. 1.

m 8. 1.

n 10. def. 1.

o 3. def. 11.

P R O P. V. Plate V. Fig. 21.

If a right-line AB be erected perpendicular to three right-lines AC, AD, AE, touching one the other at the common section, those three lines are in the same plane.

For AC, AD, (a) are in one plane FC; (a) and AD, AE, are in one plane BE, which if you conceive to be several planes, then let their intersection (b) be the right-line AG; therefore because BA by the Hyp. is perpendicular to the right-lines AC, AD, (c) and so to the plane FC, (d) it is also perpendicular to the right-line AG. therefore (since (a) that AB is in the same plane with AG, AE) the angles BAG, BAE, are right-angles, and consequently equal, the part and the whole. Which is absurd.

a 2. 11.

b 3. 11.

c 4. 11.

d 3. def. 11.

P R O P. VI. Fig. 22.

If two right-lines AB, DC, be erected perpendicular to one and the same plane EF, those right-lines AB, DC, are parallel one to the other.

Draw AD, whereunto let DG = AB be perpendicular in the plane EF, and join BD, BG, AG. Because in the triangles BAD, ADG, the angles DAB, ADG (a) are right-angles, and AB (b) = DG, and AD is common, (c) therefore BD = AG. whence in the triangles AGB, BGD, equilateral one to the other, the angle BAG is (d) = BDG; of which since BAG is a right-angle, BDG shall be so also, but the angle GDC is supposed right, therefore the right-line GD is perpendicular to the three lines DA, DB, CD, (e) which are therefore in the same plane (f) wherein AB is. Wherefore since AB

a hyp.

b constr.

c 4. 1.

d 8. 1.

e 5. 1.

f 2. 1.



g 28. 1.

and CD are in the same plane, and the internal angles BAD, CDA, are right-angles, (g) AB and CD shall be parallels. *Which was to be demonstrated.*

## P R O P. VII. Plate V. Fig. 23.

If there are two parallel right-lines AB, CD, and any points E, F, be taken in both of them, the line EF which is joined at these points, is in the same plane, with the parallels AB, CD.

a 3. 11.

b 14. ax. 1.

Let the plane in which AB, CD, are, be cut by another plane at the points E, F; then if EF is not in the plane ABCD, it shall not be the common section. Therefore let EGF be the common section; which (a) then is a right-line, therefore two right-lines EF, EGF, include a superficies. (b) *Which is absurd.*

## P R O P. VIII. Fig. 24.

If there are two parallel right-lines AB, CD, whereof one AB, is perpendicular to a plane EF, then the other CD shall be perpendicular to the same plane EF.

a 4. 11.

b 7. 11.

c 3. def. 11.

d 29. 1.

e 4. 11.

The preparation and demonstration of the sixth of this Book being transferr'd hither; the angles GDA, and GDB are right-angles: (a) Therefore GD is perpendicular to the plane, wherein are AD, DB, (b) (in which also AB, CD, are.) (c) therefore GD is perpendicular to CD; but the angle CDA is also (d) a right-angle, (e) therefore CD is perpendicular to the plane E F. *Which was to be demonstrated.*

## P R O P. IX. Fig. 25.

Right-lines (AB, CD) which are parallel to the same right-line EF, but not in the same plane with it, are also parallel one to the other.

In the plane of the parallels AB, EF, draw HG perpendicular to EF; also in the plane of the parallels EF, CD, draw IG perpendicular to EF. (a) Therefore EG is perpendicular to the plane wherein HG, GI are; and AH, CI are perpendicular to the same plane, (c) therefore AH and CI are parallels. *Which was to be demon.*

P R O P.

PROP. X. Plate V. Fig. 26.

If two right-lines AB, AC, touching one another be parallel to two other right-lines ED, DF, touching one another, and not being in the same plane, those right-lines contain equal angles, BAC, EDF.

Let AB, AC, DE, DF, be equal one to the other, and draw AD, BC, EF, BE, CF. Since AB, DE, (a) are parallels and equal, (b) also BE, AD, are parallels and equal. In like manner CF, AD, are parallel and equal; (c) therefore also BE, FC, are parallel and equal. (d) Therefore BC, EF are equal. Wherefore since the triangles BAC, EDF, are of equal sides one to the other, the angles BAC, EDF (e) shall be equal. Which was to be demonstrated.

a hyp. and  
constr.  
b 33. I.  
c 2. ax. I.  
and 9. II.  
d 33. I.  
e 8. I.

PROP. XI. Fig. 27.

From a point given on high A to draw a right-line AI perpendicular to a plane below BC.

In the plane BC draw any line DE; to which from the point A (a) draw the perpendicular AF, to the same DE through F in the plane BC (b) draw the perpendicular FH, then to FH (a) draw the perpendicular AI, this shall be perpendicular to the plane BC.

a 12. I.  
b 11. I.  
c 31. I.  
d constr.  
e 4. II.  
f 8. II.  
g 3. def. II.  
h constr.  
i 4. II.

For thro' I (c) let KIL be drawn parallel to DE. Because DE (d) is perpendicular to AF, and FH. (e) therefore DE shall be perpendicular to the plane IFA; and so also KL (f) is perpendicular to the same plane; (g) therefore the angle KIA is a right-angle, but the angle AIF is also (b) a right-angle; (i) therefore AI is perpendicular to the plane BC. Which was to be done.

PROP. XII. Fig. 28.

In a plane given BC, at a point given therein A, to erect a perpendicular line AF.

From some point D without the plane, (a) draw DE perpendicular to the said plane BC, and joining the points A, E, by a line AE, (b) draw AF parallel to DE (c) it is apparent that AF is perpendicular to the plane BC. Which was to be done.

a 11. II.  
b 31. I.  
c 8. II.

This and the preceding problem are practically performed by applying two Squares to the point given; as appears by 4. II.

PROP.

## P R O P. XIII. Plate V. Fig 29.

At a point given C in a plane given AB, two right-lines CD, CE, cannot be erected perpendicular to the said given plane on the same side.

a 6. 11.

For both CD, and CE (a) should then be perpendicular to the plane AB, and consequently parallel; which is repugnant to the definition of parallel-lines.

## P R O P. XIV. Fig. 30.

Planes CD, FE, to which the same right-line AB is perpendicular, are parallel.

hyp. and  
a def. 11.  
b 17. 1.

If you deny this; then let the planes CD, EF meet so that their common section be the right-line GH, in which take any point I, draw to it the right-lines IA, IB, in the said planes, whereby in the triangle IAB, two angles IAB, IBA (a) are right-angles. (b) Which is absurd.

## P R O P. XV. Fig. 31.

If two right-lines AB, AC, touching one the other, are parallel to two other right-lines DE, DF, touching one the other, and not being in the same plane with them, the planes BAC, EDF, drawn by those right-lines are parallel one to the other.

a 11. 11.  
b 31. 1.  
c 9. 11.  
d 3. def. 11.  
e 29. 1.  
f 4. 11.  
g constr.  
h 14. 11.

From A (a) draw AG perpendicular to the plane EF (b) and let GH, GI be parallel to DE, DF. (c) these also shall be parallel to AB, AC. Therefore since the angles, IGA, HGA, (d) are right-angles, also CAG, BAG (e) shall be right-angles; (f) therefore GA is perpendicular to the plane BC; but the same is perpendicular to the plane EF, (b) therefore the planes BC, EF, are parallel. Which was to be demonstrated.

## P R O P. XVI. Fig. 32.

If two parallel planes AB, CD, are cut by some other plane HEIGF, their common sections EH, GF are parallel one to the other.

a 1. 11.

For if they are said to be not parallel, then, since they are in the same cutting plane, they must meet somewhere, suppose in I, wherefore since the whole line HEI, FGI (a) are in the planes AB, CD, produced, the planes also shall meet. contrary to the Hyp.

†

P R O



PROP. XVII. Plate V. Fig. 33.

If two right-lines ALB, CMD, are cut by parallel planes EF, GH, IK: they shall be cut proportionally, ( $AL : LB :: CM : MD$ .)

Let the right-lines AC, BD, be drawn in the planes EF, IK; as also AD meeting the plane GH in the point N, and join NL, NM, the planes of the triangles ADC, ADB, make the sections BD, LN, and AC, NM, (a) parallels. Therefore  $AL : LB :: AN : ND$  (b) ::  $CM : MD$ . Which was to be demonstrated.

a 16. 11.  
b 2. 6.

PROP. XVIII. Fig. 34.

If a right-line AB be perpendicular to some plane CD, all the planes EF passing thro' that right-line AB shall be perpendicular to the same plane CD.

Let there be some plane BF drawn thro' AB, making the section EG with the plane CD; from some point whereof H, (a) draw HI parallel to AB in the plane EF; (b) then shall HI be perpendicular to the plane CD, and so likewise any other lines, that are perpendicular to EG; (c) therefore the plane EF is perpendicular to the plane CD; and for the same reason any other planes drawn thro' AB shall be perpendicular to CD. Which was to be demonstrated.

a 31. 1.  
b 8. 11.  
c 4. def. 11.

PROP. XIX. Fig. 35.

If two planes AB, CD, cutting one the other, are perpendicular to some plane GH, their line of common section EF shall be perpendicular to the same plane (GH.)

Because the planes AB, CD, are taken perpendicular to the plane GH, it appears by 4. def. 11. that from the point F there may be drawn in both planes AB, CD, a perpendicular to the plane GH, which shall be (a) one and the same line, and therefore the common section of the said planes. Which was to be demonstrated.

a 13. 11.

PROP. XX. Fig. 36.

If a solid angle ABCD be contained under three plane angles, BAD, DAC, BAC, any two of them howsoever taken are greater than the third.

If the three angles are equal, the assertion is evident; if unequal, then let the greatest be BAC; from whence (a) take

a 23. 1.

take away,  $BAE = BAD$ ; and make  $AD = AE$ ; and also draw  $BEC$ ,  $BD$ ,  $DC$ .

b *constr.*

c 4. 1.

d 20. 1.

e 5. *ax.* 1.

f 25. 1.

g 4. *ax.* 1.

Because the side  $BA$  is common, and  $AD$  (b)  $= AE$ ; and the angle  $BAE$  (b)  $= BAD$ , (c) thence is  $BE = BD$ ; but  $BD + DC$  is (d)  $\sqsubset BC$ ; therefore  $DC \sqsubset EC$ . Wherefore since  $AD$  (b)  $= AE$ , and the side  $AC$  is common, and  $DC \sqsubset EC$ , (f) the angle  $CAD$  shall be  $\sqsubset EAC$ , (g) therefore the angle  $BAD + CAD \sqsubset BAC$ . Which was to be demonstrated.

PROP. XXI. Plate V. Fig. 37.

Every solid angle  $A$  is contained under less angles than four plane right-angles.

a 32. 1. and

sch. 31. 1.

b 20. 11.

c 5. *ax.* 1.

For let a plane any wise cutting the sides of the solid angle  $A$  make a many-sided figure  $BCDE$ , and as many triangles  $ABC$ ,  $ACD$ ,  $ADE$ ,  $AEB$ . I denote all the angles of the polygone by  $X$ ; and I term the sum of the angles at the bases of the triangle  $Y$ ; whereof  $X + 4$  right-angles (a)  $= Y + A$ , but because that (of all the angles at  $B$ ) (b) the angle  $ABE + ABC$  is  $\sqsubset CBE$ , and the same is true also of the angles at  $C$ , at  $D$ , and at  $E$ , (c) it is manifest that  $Y$  is  $\sqsubset X$ , and consequently  $A$  shall be  $\sqsubset 4$  right-angles. Which was to be demonstrated.

PROP. XXII. Fig. 38.

If there are three plane angles  $A$ ,  $B$ ,  $HCI$ , whereof two howsoever taken are greater than the third, and the right-lines which contain them are equal  $AD$ ,  $AE$ ,  $FB$ , &c. then of the right-lines  $DE$ ,  $FG$ ,  $HI$ , connecting those equal right-lines together, it is possible to make a triangle.

a 22. 1.

b 23. 1.

c 4. 1.

d *byp.*

e 24. 1.

f 20. 1.

A triangle may be (a) made of them, if any two be greater than the third; but they are so. For (b) make the angle  $HCK = B$ , and  $CK = CH$ , and draw  $HK$ ,  $IK$ ; (c) thence  $KH = FG$ , and because the angle  $KCI$  (d)  $\sqsubset A$ ; (e) therefore  $KI \sqsubset DE$ , but  $KI$  (f)  $\sqsubset HI + KH$  ( $FG$ ;) therefore  $DE \sqsubset HI + FG$ . By the like argument any other two may be proved greater than the third; and consequently (a) it is possible to make a triangle of them. Which was to be demonstrated.

PROP.

P R O P. XXIII. Plate V. Fig. 39.

To make a solid angle MHIK of three plane angles A, B, C, whereof two howsoever taken are greater than the third. \* But it is necessary that those three angles be less than four right-angles.

Make AD, AE, BE, BF, CF, CG, equal one to the other; and of the subtended-lines DE, EF, FG (that is, of the equal lines GH, IK, KH) (a) make the triangle HKI; about which (b) describe the circle LHKI. \* But because AD is  $\perp$  HL, (c) let ADq be  $\equiv$  HLq + LMq. (d) and let LM be perpendicular to the plane of the circle HKI; and draw HM, KM, IM. Wherefore since the angle HLM (e) is a right-angle. (f) thence is MHq  $\equiv$  HLq + LMq (g)  $\equiv$  ADq. therefore MH  $\equiv$  AD. By the same way of reasoning MK, MI, AD (that is AE, AB, &c.) are equal; therefore since HM  $\equiv$  AD, and MI  $\equiv$  AE, and DE (b)  $\equiv$  HI, (k) the angle A shall be  $\equiv$  HMI, (k) as likewise the angle IMK  $\equiv$  B, (k) and the angle HMK  $\equiv$  C, wherefore a solid angle is made at M of the three given plane angles. Which was to be done. AD is assumed to be  $\perp$  HL. But this is manifest. For if AD be  $\equiv$  or  $\parallel$  HL, then is the angle A (l)  $\equiv$  (m) or  $\perp$  HLI. In like manner shall B be equal or  $\perp$  HLK, and C  $\equiv$  or  $\perp$  KLI. wherefore A + B + C \* shall either equal or exceed four right-angles, contrary to the Hyp. therefore rather let AD be  $\perp$  HL. Which was to be demonstrated.

P R O P. XXIV. Fig. 40.

If a solid AB be contained under parallel planes, the opposite planes thereof (AG, BD, &c.) are like and equal parallelograms.

The plane AC cutting the parallel planes AG, DB, (a) makes the sections AH, DC, parallels, and for the same reason AD, HC are parallels. Therefore ADCH is a pgr. By the like argument the other planes of the parallelepipedon are (b) pgrs. wherefore since AF is parallel to HG, and AD to HC, (c) the angle FAD shall be  $\equiv$  GHC, therefore because AF (d)  $\equiv$  HG, and AD (d)  $\equiv$  HC, and so AF : AD :: HG : HC, the triangles FAD, GHC (g) are like and (b) equal; and consequently pgrs. AE, HB are like and (k) equal, and the same may be shewn of the rest of the opposite planes, therefore, &c.

P R O P.

\* 21. 11.

a 22. 11.

and 21. 1.

b 5. 4.

\* See Clavius

c sch. 47. 1.

d 12. 11.

e 3. def. 11.

f 47. 1.

g constr.

h constr.

k 8. 1.

l constr. &c.

8. 1.

m 21. 1.

\* 4 cor. 13. 1.

a 16. 11.

b 35. def. 1.

c 10. 11.

d 34. 1.

e 7. 5.

g 6. 6.

h 4. 1.

k 6. ax. 1.



## P R O P. XXV. Plate V. Fig. 41.

If a solid Parallelepipedon ABCD be cut by a plane EF parallel to the opposite planes AD, BC; then as the base AH is to the base BH, so shall solid AHD be to solid BHC.

Conceive the parallelepipedon ABCD to be extended on either side, and take  $AI = AE$ , and  $BK = EB$ , and put the planes IQ, KP, parallel to the planes AD, BC; then the pgrs. IM, AH, and (a) DL, DG, (b) and IQ, AD, EF, &c. are (a) like and equal (c) wherefore the parallelepipedon AQ is  $= AF$ ; and for the same reason the parallelepipedon BP  $= BF$ ; therefore the solids IF, EP are as multiple of the solids AF, EC, as the bases IH, KH, are of the bases AH, BH. And if the base IH be  $\square, =, \sqsupset$  KH, (d) likewise shall the solid IE be  $\square, =, \sqsupset$  EP; (e) consequently  $AH : BH :: AF : EC$ . Which was to be demonstrated.

The same may be accommodated to all sorts of prisms, whence.

a 36. 1. and  
1 def. 6.

b 24. 11.

c 10. def.  
11.

d 24. 11.  
and 9. def.  
11.

e 6. def. 5.

Coroll.

If any prism whatsoever be cut by a plane parallel to the opposite planes, the section shall be a figure equal and like to the opposite planes.

## P R O P. XXVI. Fig. 42.

Upon a right-line given AB, and at a point given in it A, to make a solid angle AHIL equal to a solid angle given CDEF.

a 11. 11.

From some point F (a) draw FG perpendicular to the plane DCE, and draw the right-lines DF, FE, EG, GD, CG. Make  $AH = CD$ , and the angle  $HAI = DCE$ , and  $AI = CE$ ; and in the plane HAI make the angle  $HAK = DCG$ , and  $AK = CG$ , then erect KL perpendicular to the plane HAI, and let KL be  $= GF$ , and draw AL. Then AHIL shall be a solid angle equal to that given CDEF. For the construction of this does wholly resemble the framing of that, as will easily appear to any who examine it.

P R O P

P R O P. XXVII. Plate V. Fig. 43.

Upon a right-line given AB to describe a parallelepipedon AK, like, and in like manner situate, with a solid parallelepipedon given CD.

Of the plane angles, BAH, HAI, BAI, which are equal to FCE, ECG, FCG, (a) make the solid angle A equal to the solid angle C. Also (b) make  $FC : CE :: BA : AH$  (b) and  $CE : CG :: AH : AI$  (whence by equality (c)  $FC : CG :: BA : AI$ ) and finish the parallelepipedon AK, which shall be like to that which is given.

a 26. 11.  
b 12. 6.  
c 22. 5.

For by the construction, the Pgr. (d) BH is like to FE, and (d) HI to EG, and (d) BI to FG, and (e) so the opposites of these to the opposites of them: Therefore the six planes of the solid AK are like to the six planes of the solid CD; (f) and consequently AK, CD, are like solids. Which was to be demonstrated.

d 1. def. 6.  
e 24. 11.  
f 9. def. 11.

P R O P. XXVIII. Fig. 44.

If a solid parallelepipedon AB be cut by a plane FGCD drawn thro' the diagonal-lines DF, CG, of the opposite planes AE, HB, that solid AB shall be equally bisected by the plane FGCD.

For because DC, FG, are (a) equal and parallels, (b) the plane FGCD is a Pgr. and because (a) the Pgrs. AE, HB, are equal and like, (b) also the triangles AFD, HGC, CGB, DFE are equal and like. But the Pgrs. AC, AG, are equal and like to FB and FD; therefore all the planes of the prism FGCDAB are equal and like to all the planes of the prism FGCDHE, and (c) consequently this prism is equal to that. Which was to be demonstrated.

a 24. 1.  
b 34. 1.

c 9. def. 11.

P R O P. XXIX. Fig. 45.

Solid parallelepipedons AGHEFBCD, AGHEMLKI, being constituted upon the same base AGHE, and \* in the same height, whose insisting lines AF, AM, are placed in the same right-lines AG, FL, are equal one to the other.

\* i.e. between the parallel planes AGHE, FLKD, & so understand it in the fol.

For

a 10. def. 11  
 & 35. 1.  
 b 3. and 2.  
 ax. 1.

For (a) if from the equal prisms AFMEDI, GBLH-CK, the common prism NBMPCI be taken away, and the solid AGNEHP be added, the Parallelepipedon AGHEFBCD shall be  $\equiv$  AGHEMLKI. Which was to be demonstrated.

P R O P. XXX. Plate V. Fig. 46.

Solid parallelepipedons ADBCHEFG, ADCBIMLK being constituted upon the same base ADBC, and in the same height, whose insisting lines AH, AI are not placed in the same right-lines, are equal one to the other.

a 34. 1.

b 29. 11.

c 1. ax. 1.

For produce the right-lines HEO, GFN, and LMO, KIP; and draw AP, DO, BQ, CN; (a) then shall DC, AB, HG, EF, PQ, ON be as well equal and parallel one to the other as AD, HE, GF, BC, KL, IM, QN, PO; (b) wherefore the parallelepipedon ADCBPONQ shall be equal to either parallelepipedon ADCBHEFG, ADCBIMLK; and (c) consequently these two are equal one to the other. Which was to be dem.

P R O P. XXXI. Plate VI. Fig. 1.

\* by height  
 understand  
 the perpen-  
 dicular  
 drawn from  
 the plane of  
 the base to  
 the opposite  
 plane.

a 18. 6.

b 27. 11 &

10. def. 11.

c 30. def. 11

d hyp. and

35. 1.

e 25. 11.

f 9. 5.

g 29. 11.

h constr.

Solid parallelepipedons, ALEKGMBI, CPOHQDN being constituted upon equal bases ALEK, CP  $\omega$  O, and in the same height are equal, one to the other.

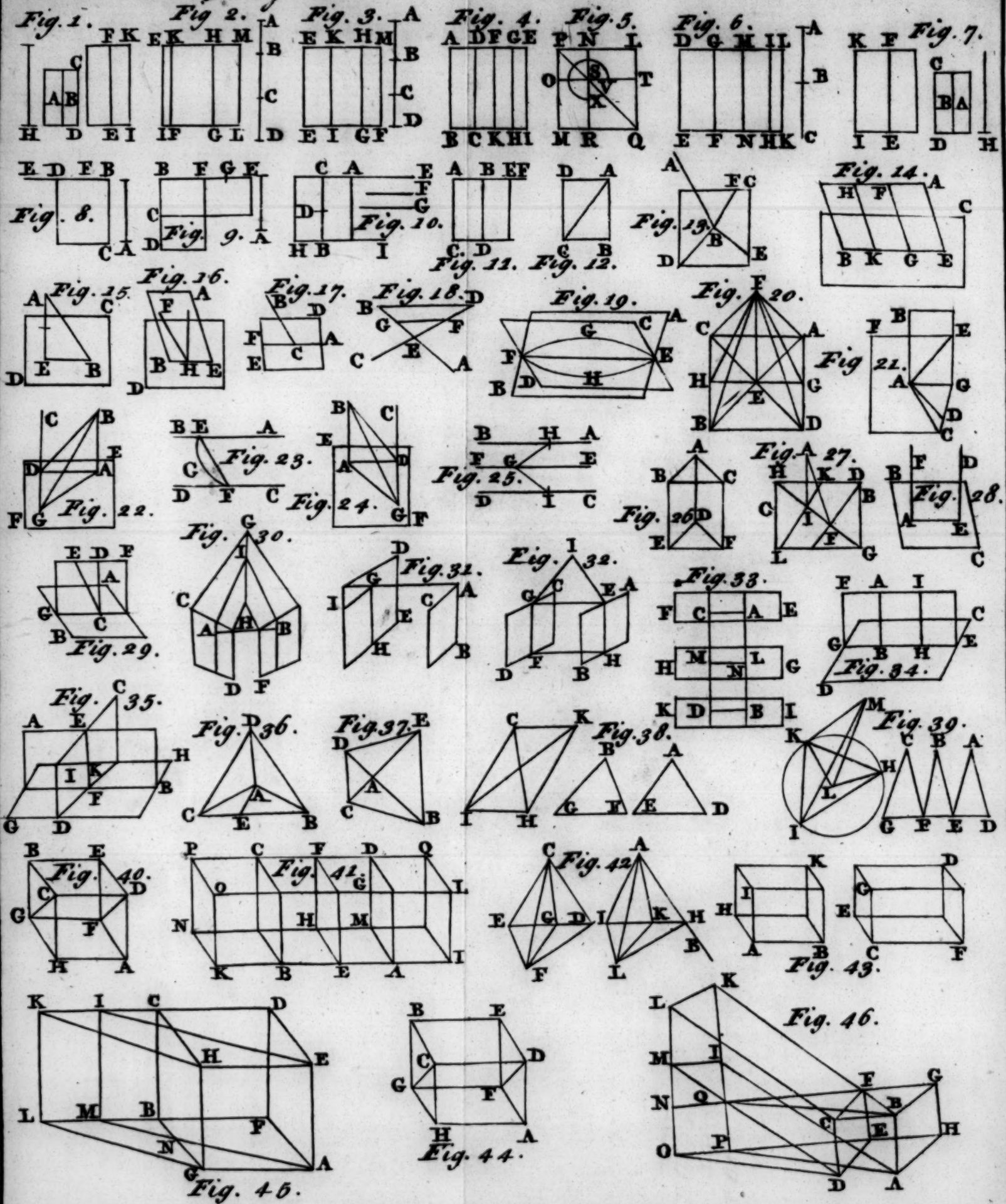
First, let the parallelepipedons AB, CD, have the sides perpendicular to the bases, and at the side C being produced, (a) make the Pgr. PRTS equal and like to the pgr. KELA. (b) and so the parallelepipedon PRTSQVYX equal and like to the parallelepipedon AB. Produce O  $\omega$  E, ND  $\delta$ ,  $\omega$  PZ, DQF, ERB,  $\delta$  V  $\gamma$ , TSZ YXF; and draw E  $\delta$ , B  $\gamma$ , ZF.

The planes O  $\epsilon$   $\delta$  N, CRVH, ZTYF, (c) are parallel one to the other; (d) and the Pgrs. ALEK, CP  $\omega$  O, PRTS, PRBZ are equal. Therefore since the parallelepipedon CD : PV  $\delta$   $\omega$  (e) :: Pgr. C  $\omega$  (PRBZ) : R  $\delta$   $\omega$  :: parallelepipedon PRBZQV  $\gamma$  F : PV  $\delta$   $\omega$ ; the parallelepipedon CD (f) shall be  $\equiv$  PRBZQV  $\gamma$  F (g) : PRVQSTYX (h)  $\equiv$  AB. Which was to be dem.

But if the parallelepipedons AB, CD, have sides oblique to the base, then on the same bases and in the same height place parallelepipedons whose sides are per-







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pendicular to the base. (*k*) They shall be equal to one another, and to those that are oblique, (*m*) whence also the oblique parallelepipeds  $AB, CD$  are equal. *Which was to be demonstrated.*

*k* 29. 11.  
*m* 1. ax. 1.

PROP. XXXII. Plate VI. Fig. 2.

Solid parallelepipeds  $ABCD, EFGH$ , of the same height, are one to the other, as their bases,  $AB, EF$ .

Produce  $EHI$ , (*a*) and make the pgr.  $FI = AB$ , and (*b*) compleat the parallelepipedon  $FINM$ . It is clear that the parallelepipedon  $FINM$ : (*c*)  $(ABCD) EFGH$  (*d*) ::  $FI (AB) : EF$ . *Which was to be demonstrated.*

*a* 45. 1.  
*b* 27. 11.  
*c* 31. 11.  
*d* 25. 11.

PROP. XXXIII. Fig. 3.

Like solid parallelepipeds,  $ABCD, EFGH$ , are to one another in triplicate ratio of their homologous sides  $AI, EK$ .

Produce the right-lines  $AIL, DIO, BIN$ , and (*a*) make  $IL, IO, IN$ , equal to  $EK, KH, KF$ , (*b*) and so the parallelepipedon  $IXMT$  equal and like to the parallelepipedon  $EFGH$ . (*c*) Let the parallepp.  $IXPB, DLYQ$  be finished. (*d*) Then shall be  $AL : IL (EK) :: DI : IO (KH) :: BI : IN (KF)$  (*e*); that is the pgrs.  $AD : DL :: DL : IX :: BO : IT$ . (*f*) i. e. the parallepp.  $ABCD :: DLYQ :: DLQY : IXBP :: IXBP : IXMT$ . (*g*)  $(EFGH)$  (*b*) therefore the proportion of  $ABCD$  to  $EFGH$  is triplicate of the proportion of  $ABCD$  to  $DLQY$ , (*h*) or of  $AI$  to  $EK$ . *Which was to be demonstrated.*

*a* 3. 1.  
*b* 27. 11.  
*c* 31. 1.  
*d* hyp.  
*e* 1. 6.  
*f* 32. 11.  
*g* constr.  
*h* 10. def. 5.  
*k* 1. 6.

Coroll.

Hence it appears that if four right-lines be continually proportional, as the first is to the fourth, so is a parallelepipedon described on the first to a parallelepipedon described on the second, being like and in like manner described.

PROP. XXXIV. Fig. 4.

In equal solid parallelepipeds  $ADCB, EHGF$ , the bases and altitudes are reciprocal ( $AD : EH :: EG : AC$ ) And solid parallelepipeds,  $ADCB, EHGF$ , whose bases and altitudes are reciprocal, are equal.

First, let the sides  $CA, GE$  be perpendicular to the bases; then if the altitudes of the solids are equal, the bases also shall be equal, and the thing is clear. But if

O

the

a 3. 1.  
 b 31. 1.  
 c 32. 11.  
 d 17. 5.  
 e 1. 6.  
 f constr.  
 g 11. 5.  
 h 32. 11.  
 k hyp.  
 l 1. 6.  
 m 32. 11.  
 n 9. 5.

the altitudes are unequal, from the greater EG (a) take EI = AC, and at I (b) draw the plane IK parallel to the base EH. Then

1. Hyp. AD : EH (c) :: parallepp. ADCB : EHIK (d) :: parallepp. EHGF : EHIK (c) :: GL : IL (e) :: GE : IE (f) (AC) ; it is plain therefore (g) that AD : EH :: GE : AC. Which was to be demonstrated.

2. Hyp. ADCB : EHIK (b) :: AD : EH (k) :: EG : EI (l) :: GL : IL (m) :: parallepp. EHGF : EHIK ; (n) wherefore the parallelepipedon ADCB = EHGF. Which was to be demonstrated.

Moreover, let the sides be oblique to the bases, and erect right parallelepipedons upon the same bases with the same altitude ; the oblique parallepp. shall be equal to them. Wherefore since by the first part, the bases and altitudes of those are reciprocal, the bases and altitudes of these also shall be reciprocal. Which was to be demon.

## Coroll.

All that hath been dem. of parallepp. in the 29, 30, 31, 32, 33, 34. Prop. does also agree to triangular prisms, which are half parallepp. as appears by Prop. 28. Therefore,

1. Triangular prisms are of equal height with their bases.

2. If they have the same or equal bases and the same altitude, they are equal.

3. If they are like, their proportion is triplicate of that of their homologous sides.

4. If they are equal, their bases and altitudes are reciprocal ; and if their bases and altitudes are reciprocal, they are also equal.

## P R O P. XXXV. Plate VI. Fig. 5.

If there are two plane angles BAC, EDF, equal, and from the point of those angles two right-lines AG, DH, elevated on high, containing equal angles with the lines first given, each to his correspondent angle (the angle GAB = HDE, and GAC = HDE) and if in those elevated lines AG, DH, some points be taken, G, H ; and from these points perpendicular lines GI, HK, drawn to the planes BAC, EDF, in which the angles first given are, and right-lines AD, DK, be drawn to the angles first given from the points I, K which are made by the perpendiculars in the planes ; the right-lines with the elevated lines AG, DH, shall contain equal angles GAM, HDK.

Make DH, AL, equal; and GI, LM parallels, and MC to AC, MB to AB, KF to DF, KE to DE perpendicular; and draw the right-lines BC, LB, LC, and EF, HF, HE; (a) and LM perpendicular to the plane BAC; (b) wherefore the angles LMC, LMA, LMB, and for the same reason the angles HKF, HKD, HKE, are right-angles. Therefore  $ALq(c) = LMq + AMq$   $(c) = LMq + CMq + ACq(c) = LCq + ACq$ ; (d) therefore the angle ACL is a right-angle. Again  $ALq(e) = LMq + MAq(e) = LMq + BMq + BAq(e) = BLq + BAq$ ; (d) therefore the angle ABL is also a right-angle. By the like inference the angles DFH, DEH are right-angles; (f) therefore  $AB = DE$ , (f) and  $BL = EH$ , (f) and  $AC = DF$ , and  $CL = FH$ ; (g) wherefore also  $BC = EF$ ; (g) and the angle  $ABC = DEF$ , (g) and the angle  $ACB = DFE$ ; (b) whence the other right-angles CBM, BCM, are equal to the other FEK, EFK (k) therefore  $CM = FK$ , (l) and so also  $AM = DK$ ; therefore if from  $LAq(m) = HDq$  be taken away  $AMq = DKq$ , (n) there remains  $LMq = HKq$ ; wherefore the triangles LAM, HDK are equilateral one to the other; (p) therefore the angle  $LAM = HDK$ . Which was to be demonstrated.

Coroll.

Therefore, if there be two plane angles equal, from those points equal right-lines are elevated on high containing equal angles with the lines first given, each to each; perpendiculars drawn from the extreme points of those elevated lines to the planes of the angles first given, are equal one to the other, viz.  $LM = HK$ .

PROP. XXXVI. Plate VI. Fig. 6.

If there are three right-lines DE, DG, DF proportional, the solid parallelepiped DH, made of them, is equal to the solid parallelepiped IN made of the middle-line DG (IL) which is also equilateral, and equiangular to the said parallelepiped DH.

Because  $DE : IK(a) :: IL : DF$ , (b) the pgr. LK will be  $= FE$ , and by reason of the equality of the plane angles at D and I, and of the lines GD, IM, the altitudes of the parallelepipeds are equal by the preceding Coroll. (c) therefore the parallelepipeds are equal to the other. Which was to be demonstrated.

a 8. 11.  
b 3. def. 11.

c 47. 1.  
d 48. 1.  
e 47. 1.

f 26. 1.  
g 4. 1.

h 3. ax. 1.  
k 26. 1.  
l 47. 1.  
m constr.  
n 47. 1. &  
3. ax.  
p 8. 1.

a hyp.  
b 14 6.

c 31. 11.



## P R O P. XXXVII. Plate VI. Fig. 7.

If there are four right-lines A, B, C, D, proportional, the solid parallelepps. A, B, C, D being like, and in like sort described from them, shall be proportional. And if the solid parallelepps. being like and in like sort described, be proportional ( $A : B :: C : D$ .) then those right-lines A, B, C, D, shall be proportional.

a 33. 11.  
b sch. 23. 5.

For the proportions of the parallelepps. (a) are triplicate of those lines; therefore if  $A : B :: C : D$ , (b) then shall the parallelep. A : parallelep. B :: parallelep. C : parallelep. D. and so also contrarily.

## P R O P. XXXVIII. Fig. 8.

If a plane AB be perpendicular to a plane AC, and a perpendicular-line EF be drawn from a plane E in one of the planes (AB) to the other point AC, that perpendicular EF shall fall upon the common section of the planes AD.

a 12. 1.  
b 4. and 3.  
def. 11  
c 17. 1.

If it be possible, let F fall without the intersection AD, and in the plane AC, (a) draw FG perpendicular to AD, and join EG. The angle FGE (b) is a right-angle, and EFG is supposed to be such also; therefore the right-angles are in the triangle EFG. (c) Which is absurd.

## P R O P. XXXIX. Fig. 9.

If the sides (AE, FC, AF, EC, and DH, GB, DG, HC) of the opposite planes AC, DB, of a solid parallelepipedon be divided into two equal parts, and planes ILQO, PKMN be drawn thro' their sections, the common section of planes ST, and the diameter of the solid parallelepipedon AB shall divide one the other into two equal parts.

a 34. 1.  
b 29. 1.  
c 4. 1.  
d sch. 15. 1.  
e 34. 1.  
f 9. 11. and  
1 ax.  
g 33. 1.  
h 7. 1. 11

Draw the right-lines SA, SC, TD, TB. Because the sides DO, OT are equal to the sides BQ, QT, and the alternate-angles TOD, TQB equal, also (c) the bases DT, TB, and the angles DTO, BTQ are equal, therefore DTB is a right-line, and so in like manner ASC. Moreover (e) as well AD is parallel and equal to FG (e) as FG to CB, and (f) thence AD is parallel to CB; (g) and consequently AC to DB (b) wherefore AB and ST are in the same plane ABCD. Therefore since the vertical-angles AVS, BVT, and the angles

...re-angles ASV, BTV are equal (*k*) and  $AS = BT$ ; *k* 7. ax. 1.  
 ...erefore shall AV be = BV, (*l*) and  $SV = VT$ . Which l 25. 1.  
 ...as to be dem.

Coroll.

Hence in every parallelepipedon, all the diameters  
 sect one another in one point, V.

PROP. XL. Plate VI. Fig. 10.

If two prisms ABCFED, GHMLIK, be of equal alti-  
 tude, whereof one hath its base ABCF a parallelogram, and  
 the other GHM a triangle, and if the parallelogram ABCF  
 be double to the triangle GHM; those prisms ABCFED,  
 GHMLIK are equal.

For if the parallelepipedons AN, GQ, be compleated,  
 they shall be equal, because of the equality (*b*) of the  
 bases AC, GP, and (*c*) of the altitudes; (*d*) therefore also  
 the prisms, (*e*) the halves thereof shall be equal. Which  
 is to be dem.

a 31. 11.  
 b 34. 1. &  
 7. ax.  
 c hyp.  
 d 28. 11.  
 e 7. ax. 1.

Schol.

From the preceding demonstrations, the demension of trian-  
 gular prisms, and quadrangular, or parallelepipedons, is learnt;  
 2. by multiplying the altitude into the base.

As if the altitude be 10 foot, and the base 100 square  
 foot (the base may be measured by sch. 35. 1. or by 41. 1.)  
 then multiply 100 by 10, and 1000 cubic foot shall be  
 produced for the solidity of the prism given.

For as a rectangle, so also is a right parallelepipedon produ-  
 ced from the altitude multiplied into the base. There-  
 fore every parallelepipedon is produced from the altitude  
 multiplied into the base, as appears by 31. of this Book.  
 Moreover, since the whole parallelepipedon is produced  
 from the altitude drawn into the base, the half thereof  
 (that is, a triangular prism) shall be produced from the  
 altitude drawn into half the base, namely the triangle.

Andr Tacq

*An Advertisement.*

Obf. That of those letters which denote a solid angle, the first is always at the point in which the angle is; but of those letters which denote a pyramid, the last is at the supreme point thereof.

Ex. gr. the solid angle ABCD is at the point A; and the supreme point of the pyramid BCDA is at the point A, and the base is the triangle BCD.

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*The End of the Eleventh Book.*

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T H E



# The TWELFTH BOOK OF EUCLID E's ELEMENTS.

PROPOSITION I. Plate VI. *Fig. 11, 12.*

**L**IKE polygonous figures *ABCDE*, *FGHIK* inscribed in circles *ABD*, *FGI*, are one to another, as the squares described on the diameters of the circles *AL*, *FM*.

Draw *AC*, *BL*, *FH*, *GM*. Because (a) the angle  $\angle ABC = \angle FGH$ , (a) and  $AB : BC :: FG : GH$  (b) therefore shall the angle  $\angle ACB$  (c) ( $\angle ALB$ ) be  $\angle FHG$  (c) ( $\angle FMG$ .) but the angles  $\angle ABL$ ,  $\angle FGM$  (d) are right and so equal ; (e) therefore the triangles *ABL*, *FGM* are equiangular. (f) wherefore  $AB : FG :: AL : FM$ . (g) therefore  $ABCDE : FGHIK :: ALq : FMq$ .

a 1. def. 6.  
b 6. 6.  
c 21. 3.  
d 31. 3.  
e 32. 3.  
f cor. 4. 6.  
g 22. 6.

*Coroll.*

Hence (because  $AB : FG :: AL : FM :: BC : GH$  & c.) the ambits of like polygonous figures inscribed in a circle are in the same (b) proportion as the diameters.

h 1. 12. &  
12. 5.

PROP. II. *Fig. 13, 14.*

Circles *ABT*, *EFN*, are in proportion one to another, as the squares of their diameters *AC*, *EG* are.

Suppose  $ACq : EGq ::$  the circle *ABT* : *I*. I say then *I* is equal to the circle *EFN*.

For first, if it be possible, let *I* be less than the circle *EFN*, and let *K* be the excess or difference. Inscribe the square *EFGH* in the circle *EFN*, (a) it being the half of a circumscribed square, and so greater than the semicircle. [(b) Divide equally in two the arches

a sch. 7. 4.  
b 30. 3.

O 4

EF

c sch. 27. 3.  
d 41. 1.

e 1. 10.

f hyp. and  
3. ax.  
g 30. 3. &  
1. post. 1.  
h 1. 12.  
k hyp.  
l 9. ax. 1.  
m 14. 5.

n hyp.

o 14. 5.  
p 11. 5.

EF, FG, GH, HE, and at the points of the divisions join the right-lines EL, LF, &c. thro' L draw the tangent PQ (c) which is parallel to EF; and produce HEP, GFQ, then is the triangle ELF (d) the half of the pgr. EPQF, and so greater than the half of the segment ELF; and in like manner the rest of those triangles exceed the halves of the rest of the segments. And if the arches EL, LF, FM, &c. be again bisected, and the right-lines joined, the triangles will likewise exceed the half of the segments, Wherefore if the square EFGH be taken from the Circle EFN, and the triangles from the other segments, and this to be done continually, at length (e) there will remain some magnitude less than K. Let us have gone so far, namely, to the segments EL, LF, FM, &c. taken together less than K. Therefore I (f) (the circle EFN — K)  $\supset$  the polyg. ELFMGNHO (the circle EFN — the segment EL + LF, &c.) In the circle ABT (g) conceive a like polygon AKBSCTDV inscribed. Therefore since AKBSCTDV : ELFMGNHO (h) :: ACq : EGq (i) :: the circle ABT : I. and the polyg. AKBSCTDV (l)  $\supset$  the circle ABT. the polyg. ELFMGNHO (m) shall be  $\supset$  I. but before, I was  $\supset$  ELFMGNHO, which is repugnant.

Again, if it be possible, let I be  $\supset$  the circle EFN. Therefore because ACq : EGq (n) :: the circle ABT : I; and inversely I : the circle ABT :: EGq : HCq. suppose I : the circle ABT :: the circle EFN : K. (o) therefore the circle ABT  $\supset$  K. (p) and EGq : ACq :: the circle EFN : K, which was just now shewn to be repugnant.

Therefore it must be concluded, that I is  $\equiv$  to the circle EFN. Which was to be demonstrated.

Coroll.

Hence it follows, that as a circle is to a circle, so is a polygon inscribed in the first to a like polygon inscribed in the second.

PROP. III. Plate VI. Fig. 15.

Every Pyramid ABCD having a triangular base, may be divided into two pyramides AEGH, HIKC, equal, and like one to the other, having bases triangular, and like to the whole ABDC; and into two equal prisms, BEGEIH, FGDIHK; which two prisms are greater than the half of the whole pyramid ABDC.

Divide

Divide the sides of the pyramid into two parts at the points E, F, G, H, I, K, and join the right-lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Because the sides of the pyramid are proportionally cut (*a*) thence HI, AB, and GF, AB; and IF, DC; and HG, DC, &c. are parallels, and consequently HI, FG; and GH, FI are also parallels, therefore it is apparent that the triangles ABD, AEG, EBF, FDG, HIK, (*b*) are equiangular, and that the four last are (*c*) equal: In like manner the triangles ACB, AHE, EIB, HIC, FGK are equiangular; and the four last are equal one to the other. Also the triangles BFI, FDK, IKC, EGH; and lastly, the triangles AHG, GDK, HKC, EFI are like and equal. Moreover the triangles, HIK to ADB, EGH to BDC, and EFI to ADC, and FGK to ABC, (*d*) are parallel. From whence it evidently follows, first, that the pyramids AEGH, HIKC are equal, and (*e*) like to the whole ABDC, and to one another. Next, that the solids BFGGEIH, FGDIHK are prisms, and that of equal heights, as being placed between the parallel planes ABD, HIK, but the base BFGE is (*f*) double of the base FDG, wherefore the said prisms are equal; whereof the one BFGGEIH is greater than the pyramid BEFI, that is, than AEGH, the whole than its part; and consequently the two prisms are greater than the two pyramids and so exceed the half of the whole pyramid ABDC. Which was to be dem.

P R O P. IV. Fig. 16, 17.

If there are two pyramids ABCD, EFGH, of the same altitude, having triangular bases ABC, EFG; and either of them be divided into two pyramids (AILM, MNOD; and EPRS, STVH) equal one to the other and like to the whole; and into two equal prisms (IBKLMN, KLCNMO; and PFQRST, QRGTSV;) and if in like manner either of these pyrs. made by the former division be divided, and this be done continually; then as the base of one pyramid is to the base of the other pyramid, so are all the prisms which are in one pyramid, to all the prisms which are in the other pyramid, being equal in multitude

For (applying the construction of the precedent prop.) BC:KC (*a*)::FG:QG; (*b*) therefore the triangle ABC is to the like triangle LKC as EFG is to (*c*) the like RQG; therefore by permutation ABC:EFG (*d*)::LKC

a 2. 6.

b 29. 1.

c 26. 1.

d 15. 11.

e 10. def.  
11.

f 2 ax. 1.

g 40. 11.

a 15. 5.

b 22. 6.

c 2. 6. &c.

d 16. 5.



*esch.* 34. 11. LKC : RQG (*e*) : : the prism KLCNMO : QRGTSV  
*f* 1. 5. (for these are of equal altitude) (*f*) : : IBKLMN : PF-  
*g* 12. 5. QRST; (*g*) wherefore the triangle ABC : EFG : : the  
 prism KLCMNO + IBKLMN : the prism QRGTSV +  
 PFQRST. Which was to be dem.

But if the pyramids MNOD, AILM; and EPRS, STVH be further divided; in like manner the four new prisms made hereby shall be to four produced before, as the bases MNO and AIL are to the bases STV, and EPR, that is, as LKC to RQG, or as ABC to EFG. (*b*) wherefore all the prisms of the pyramid ABCD are to all the prisms of the pyramid EFGH as the base ABC is to the base EFG. Which was to be dem.

P R O P. V. Plate VI. Fig. 18, 19.

Pyramids ABCD, EFGH, being of the same altitudes and having triangular bases ABC, EFG, are one to another as their bases ABC, EFG.

Let the triangle ABC : EFG : : ABCD : X. I say X is equal to the pyramid EFGH. For if it be possible, let X be  $\sqsupset$  EFGH. and let the excess be Y. Divide the pyramid EFGH into prisms and pyramids, and the other pyramids in like manner, (*a*) till the pyr. left EPRS, STVH, be less than the solid Y. Therefore since the pyramid EFGH = X + Y, it is manifest that the remaining prisms PFQRST, QRGTSV are greater than the solid X. Conceive the pyramid ABCD divided after the same manner; (*b*) then will be the prism IBKLMN + KLCNMO : PFQRST + QRGTSV : : ABC : EFG (*c*) : : the pyr. ABCD : X; (*d*) therefore X  $\sqsubset$  the prism PFQRST + QRGTSV; which is contrary to that which was affirmed before.

Again, conceive X  $\sqsubset$  the pyr. EFGH, and make the pyr. EFGH : Y : : X : the pyr. ABCD (*e*) : : EFG : ABC. Because EFGH (*f*)  $\sqsupset$  X, (*g*) thence Y  $\sqsupset$  the pyr. ABCD; which is shewn before to be impossible. Therefore I conclude, that X is equal to the pyr. EFGH. Which was to be demonstrated.

P R O P.

P R O P. VI. Plate VI. Fig. 20, 21.

Pyramids  $ABCDEF$ ,  $GHIKLM$ , being of the same altitude, and having polygonous bases  $ABCDE$ ,  $GHIKL$ , are to one another as their bases  $ABCDE$ ,  $GHIKL$ . are,

Draw the right-lines  $AC$ ,  $AD$ ,  $GI$ ,  $GK$ , then is the base  $ABC : ACD$  ( $a$ ) : : the pyr.  $ABCF : ACDF$ ; ( $b$ ) therefore by composition,  $ABCD : ACD : : \text{the pyr. } ABCDF : ACDF$ ; ( $a$ ) but also  $ACD : ADE : : \text{the pyr. } ACDF : ADEF$ ; ( $c$ ) therefore by equality  $ABCD : ADE : : ABCDF : ADEF$ , and ( $b$ ) thence by composition  $ABCDE : ADE : : \text{the pyr. } ABCDEF : ADEF$ ; moreover  $ADE : GKL$  ( $d$ ) : : the pyr.  $ADEF : GKLM$ ; and as before, and inversely  $GKL : GHIKL : : \text{the pyr. } GKLM : GHIKLM$ ; ( $e$ ) therefore again by equality  $ABCDE : GHIKL : : \text{the pyr. } ABCDEF : GHIKLM$ . Which was to be demonstrated.

If the bases have not sides of equal multitude, the demonstration will proceed thus. The base (Fig. 20, 22.)  $ABC : GHI$  ( $e$ ) : : the pyr.  $ABCF : GHIK$  ( $e$ ) and  $ACD : GHI : : \text{the pyr. } ACDF : GHIK$ , ( $f$ ) therefore the base  $ABCD : GHI : : \text{the pyr. } ABCDF : GHIK$ . ( $e$ ) Moreover the base  $ADE : GHI : : \text{the pyr. } ADEF : GHIK$ , ( $f$ ) therefore the base  $ABCDE : GHI : : \text{the pyr. } ABCDEF : GHIK$ .

P R O P. VII. Fig. 23.

Every prism,  $ABCDEF$ , having a triangular base, may be divided into three pyrs.  $ACDF$ ,  $ACBF$ ,  $CDFE$ , equal one to the other, and having triangular bases.

Draw the diameters of the parallelograms  $AC$ ,  $CF$ ,  $FD$ . Then the triangle  $ACB$  is ( $a$ )  $= ACD$ ; ( $b$ ) therefore the pyramids of equal height  $ACBF$ ,  $ACDF$ , are equal. In like manner the pyr.  $DFAC = \text{the pyr. } DFEC$ , but  $ACDF$  and  $DFAC$  are one and the same pyr. ( $c$ ) therefore the three pyramids  $ACBF$ ,  $ACDF$ ,  $DFEC$ , into which the prism is divided, are equal one to the other. Which was to be demonstrated.

Hence, every pyramid is the third part of the prism that has the same base and height with it, or every prism is treble of the pyramid that has the same base and height with it.

For

a 5. 12.  
b 18 5.  
c 22. 5.

d 5. 12.

e 5. 12.

f 24. 5.

a 34. 11  
b 5. 12.  
c 1. ax. 1.

a 7. 12.

b 1. 5.

For resolve the polygonous prism  $ABCDEFGHIK$ , (Fig. 24.) into triangular prisms; and the pyr.  $ABCD$ .  $EH$  into triangular pyramids; (a) then all the parts of the prism shall be treble to all the parts of the pyramids, (b) consequently the whole prism  $ABCDEFGHIK$  is treble to the whole pyr.  $ABCDEH$ . Which was to be demonstrated.

27. 2 8

28. 2 8

29. 2 8

P R O P. VIII. Plate VI. Fig. 25, 26.

Like pyramids  $ABCD$ ,  $EFGH$ , which have triangular bases  $ABC$ ,  $EFG$  are in triplicate ratio of their homologous sides  $AC$ ,  $EG$ .

a 27. 11.

b 9. def. 11.

c 28. 11.

and 7. 12.

d 15. 5.

e 33. 11.

(a) Compleat the parallelps.  $ABICDMKL$ ,  $EFNG$ .  $HQOP$ , which (b) are like, and (c) sextuple of the pyramids  $ABCD$ ,  $EFGH$ ; (d) and therefore the pyrs. have the same proportion to one another as the parallelps. have, that is, (e) triplicate of their homologous sides.

Coroll.

21. 2 9

Hence, also like polygonous pyramids are in triplicate ratio of their homologous sides; as may be easily prov'd by resolving them into triangular pyramids.

2. 4 3

P R O P. IX. Fig. 25, 26.

In equal pyramids  $ABCD$ ,  $EFGH$ , having triangular bases  $ABC$ ,  $EFG$ , the bases and altitudes are reciprocal; And pyramids having triangular bases, whose altitudes and bases are reciprocal, are equal.

a 28. 11.

and 7. 12.

b 34. 11.

c 15. 5.

d 15. 5.

e 15. 5.

f 34. 11.

g 6. ax. 1.

1. Hyp. The compleated parallelps.  $ABICDMKL$ ,  $EFNGHQOP$  are (a) sextuple of the equal pyramids  $ABCD$ ,  $EFGH$  (each to each) and so equal one to the other; therefore as the altitude (H) : the altitude (D) (b) ::  $ABIC$  :  $EFNG$ ; (c) ::  $ABC$  :  $EFG$ . Which was to be demonstrated.

2. Hyp. The altitude (H) : the altitude (D) (d) ::  $ABC$  :  $EFG$  (e) ::  $ABIC$  :  $EFNG$  (f) therefore the parallelps.  $ABICDMKL$ ,  $EFNGHQOP$  are equal; (g) consequently also the pyramids  $ABCD$ ,  $EFGH$  being subsextuple of the same, are equal. Which was to be demonstrated.

The same is applicable to polygonous pyramids, for they may also in like manner be reduced to triangular.

Coroll



Coroll.

Whatever is dem. of pyramids in prop. 6, 8, 9 does likewise agree to any sort of prisms; seeing they are triple of the pyramids that have the same base and altitude with them. Therefore

1. The proportion of prisms of equal altitude is the same with that of their bases.
2. The proportion of like prisms is triplicate of that of their homologous sides.
3. Equal prisms have their bases and altitudes reciprocal; and prisms which are so reciprocal, are equal.

Schol.

From what has been hitherto dem. the dimension of any prisms and pyramids may be collected.

(a) The solidity of a prism is produced from the altitude multiplied into the base; (b) and therefore likewise that of a pyr. from the third part of the altitude multiplied into the base.

a<sup>1</sup>. cor. 12.

ℰ sch. 4.

11.

b 7. 12.

PROP. X. Plate VI. Fig. 27.

Every Cone is the third part of a cylinder having the same base with it ABCD, and the altitude equal.

If you deny it, then first let such cylinder be more than triple to the cone, and let the excess be E. A prism described on a square in the circle ABCD (a) (Fig. 13, 14.) is subduple of a prism described upon a square about the circle, being equal to it and the cylinder in height. Therefore a prism upon the square, ABCD exceeds the half of the cylinder, and likewise a prism upon the base AFB, of equal height to the cylinders, (b) is greater than the half of the segment of the cyl. AFB, continue an equal bisection of the arches, and subtract the prisms till the remaining segments of the cyl. namely, at AF, FB, &c. become less than the solid E. Therefore the cyl. — segm. AF, FB, &c. (the prism on the base AFBGCH-DI) (c) is greater than the cyl. — E (d) (the triple of the cone) therefore the pyr. (e) a third part of the said prism (being placed on the same base, and of the same height) is greater than the cone of equal height on the base ABCD a circle, i.e. the part greater than the whole Which is absurd.

a sch. 7. 4.

and cor. 9.

12.

b sch. 27. 3.

and cor. 9.

12.

c 5. ax. 1.

d hyp.

e cor. 7. 12.

But

But if the cone be affirmed to be greater than the third part of the cyl. then let the excess be E. Subtract the pyrs. from the cone, as you did in the first part the prisms from the cyl. till some segments of the cone remain, suppose AF, FB, BG, &c. less than the solid E, therefore the cone — E ( $f$ )  $\frac{1}{3}$  of the cyl.)  $\supset$  the pyr. AFBGCHDI (the cone — seg AF, FB, &c.) therefore the prism triple to the pyr. (*viz.* of equal height, and on the same base) is greater than the cyl. on the base ABCD, the part than the whole. *Which is absurd.* Wherefore it must be granted, that the cyl. is equal to triple of the cone. *Which was to be dem.*

P R O P. XI. Plate VI. Fig. 28. 29.

*Cylinders and Cones ABCDK, EFGHM, being of the same altitude, are to one another as their bases ABCD, EFGH are.*

Let the circle ABCD : the circle EFGH :: the cone ABCDK : N. I say N is equal to the cone EFGHM.

For if it be possible, let N be  $\supset$  the cone EFGHM, and let the excess be O. The preparation and argumentation of the prec. prop. being supposed ; then shall O be greater than the segments of the cone EP, PF, FQ, &c. and so the solid N  $\supset$  the pyr. EPFQGRHSM. In the circle ABCD (*a*) make a like polyg. fig. ATBVXDY. Because the pyr. ABVYK : the pyr. EFQSM (*b*) :: the polygon ATBVY : the polygon EPFQS (*c*) :: the circle ABCD : the circle EFGH (*d*) :: the cone ABCDK : N (*e*) thence the pyr. EPFQGRHSM shall be  $\supset$  N, contrary to what was affirmed before. Again, conceive N  $\supset$  the cone EFGHM, and make the cone FFGHM : O :: N : the cone ABCDK (*f*) :: the circle EFGH : ABCD ; (*g*) therefore O  $\supset$  the cone ABCDK ; *which is absurd*, as appears by what is shewn in the first part.

Therefore rather admit ABCD : EFGH :: the cone ABCDK : EFGHM. *Which was to be dem.*

The same may be dem. of cylinders, if cylinders and prisms be conceived in the place of cones and pyramids, therefore, &c.

*Schol.*

Hence, is gathered the dimension of all sorts of cylinders and cones. The solidity of a right cyl. is produced from the circular base (*a*) (the dimension whereof is to be learnt out of Archimedes) multiplied into the height ; (*b*) whence in like manner that of every cylinder.

(*c*) Therefore

a 30. 3. and  
i post.  
b 6. 12.  
c cor. 2. 12.  
d hyp.  
e 14. 5.  
f hyp. and by  
inversion.  
g 14. 5.

a 1. prop de  
dimension. cir.  
b 11. 12.

(*c*)  
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(c) Therefore the solidity of a cone is produced from the third part of the altitude multiplied into the base. e 10. 12.

PROP. XII. Plate VI. Fig. 30, 31.

Like cones and cylinders ABCDK, EFGHM, are in triplicate ratio of that of the diameters TX, PR, of their bases ABCD, EFGH.

Let the cone A have to N a triplicate ratio of TX to PR. I say N is = the cone EFGHM. For if it be possible let N be  $\supset$  EFGHM, and let the excess be O, therefore N  $\supset$  the pyr. EPFQGRHSM. Let the axes of the cones be IK, LM, and join the right-lines VK, CK, VI, CI, and QM, GM, QL, GL. Because the cones are like, (a) thence VI : IK :: QL : LM; but the angles VIK, QLM (b) are right-angles, (c) therefore the triangles VIK, QLM are equiangular, (d) whence VC : VI :: QG : QL. also VI : VK :: QL : QM. therefore by equality VC : VK :: QG : QM. (e) moreover VK : CK :: QM : MG; therefore again by equality VC : CK :: QG : GM; (f) therefore the triangles VKC, QMG are like; and by a like way of reasoning the other triangles of this pyr. are like to the other of that, (g) wherefore the pyrs. themselves are like. (b) But these are in triplicate proportion of that of VC to QG, (h) that is, of VI to QL, (i) or TX to PR; (m) therefore the pyr. ATBVCXDYK : the pyr. EPFQGRHSM :: the cone ABCDK : N; (n) whence the pyr. EPFQGRHSM  $\supset$  N. which is repugnant to what was affirmed before.

Again, take N  $\supset$  the cone EFGHM, make the cone EFGHM : O :: N : the cone ABCDK (o) :: the pyr. EPRM : ATCK (p) :: GQ : VC thrice :: (q) PR : TX thrice, therefore O (r) is  $\supset$  ABCDK. which was before shewn to be repugnant. Wherefore N = the cone EFGHM. Which was to be demonstrated.

But forasmuch as what proportion soever cones have, also cylinders, being triple of them, have the same; therefore cyl. shall be to cyl. in triplicate ratio of the diameters of their bases.

a 24. def. 11.  
b 18. def. 11.  
c 6. 6.  
d 4. 6.  
e 7. 5.  
f 5. 6.

g 9. def. 11.  
h cor. 8. 12.  
k 4. 6.  
l 15. 5.  
m hyp. and  
11. 5.  
n 14. 5.

o before &  
in-versly.  
p cor. 8. 12  
q 4. 6.  
r 14. 5.

PROP.



## P R O P. XIII. Plate VI. Fig 32.

If a cyl. ABCD be divided by a plane EF parallel to the opposite planes BC, AD, then as one of cyl. AEFD is to the other cyl. EBCF, so is the axis GI to the axis IH.

a 3. 1.

b 11. 12.

c 11. 12.

d 6. def. 5.

The axis being produced, (a) take  $GK = GI$ , and  $HL = IH = LM$ , and conceive planes drawn at the points K, L, M, parallel to the circles AD, BC, (b) therefore the cyl. ED = the cyl. AN, and the cyl. EC (b) = BO (b) = OP; therefore the cyl. EN is the same multiple of the cyl. ED as the axis IK is of the axis IG, and in like manner the cyl. EP is the same multiple of the cyl. BF, as the axis IM is of the axis IH; but as  $IK = IM$ , (c) so is the cyl. EN = EP (d) therefore the cyl. AEFD : the cyl. EBCF :: GI : IH. Which was to be dem.

## P R O P. XIV. Fig. 33.

Cones AEB, CFD, and cylinders AH, CK, standing upon equal bases AB, CD, are to one another, as their altitudes ME, NF

a 11. 12.

b 13. 12.

\* apply 9. & 7. 12.

The cyl. HA, and the axis EM, being produced, take  $ML = FN$ ; and thro' the point L draw a plane parallel to the base AB, (a) then shall the cyl. AP be = CK, (b) but the cyl. AH : AP (CK) :: ME : ML (NF.) Which was to be dem.

The same may be affirmed of cones which are subtriple of cylinders; \* as also of prisms and pyramids.

## P R O P. XV. Fig. 34.

In equal cones BAC, EDF and cylinders BH, EK, the bases and altitudes are reciproc. ( $BC : EF :: MD : LA$ ) And cones and cylinders, whose bases and altitudes are reciprocal, are equal one to the other.

If the altitudes be equal then the bases are equal too, and the thing is evident. If unequal, then take away  $MO = LA$ .

a 14. 12.

b constr.

c hyp.

d 11. 12.

1. Hyp. Then is  $MD : MO$  (a) (LA) (b) :: the cyl. EK : (c) (BH) EQ (d) :: the cir. BC : EF. Which was to be dem.

2. Hyp.

2. Hyp.  $BC : EF (e) :: DM : OM (LA) (f) ::$  the  
cyl.  $EK : EQ (g) :: BC : EF (b) :: BH : EQ. (k)$  There-  
fore the cyl.  $EK :: BH$ . Which was to be dem.  
The same argument may be used for cones.

e hyp.  
f 14. 12.  
g 11. 5.  
h 11. 12.  
k 9. 5.

PROP. XVI. Plate VI. Fig. 35.

Two unequal circles  $ABCG, DEF$ , having the same cen-  
ter  $M$ , to inscribe in the greater circle  $ABCG$  a polygonous  
figure of equal and even sides, which shall not touch the  
lesser circle  $DEF$ .

Thro' the center  $M$  draw the line  $AC$  cutting the cir-  
cle  $DEF$  in  $F$ , from whence raise a perpendicular  $FH$ , (a)  
divide the semicircle  $ABC$  into two equal parts; and  
the half thereof  $BC$  also; and so do continually (b) till  
the arch  $IC$  becomes less than the arch  $HC$ ; from  $I$  let  
fall the perpendicular  $IL$ . It is manifest that the arch  
 $IC$  measures the whole circle; and that the number of  
arches is even, and so that the subtended-line  $IC$  is the  
side (c) of the polygon that may be inscribed without  
touching the lesser circle  $DEF$ . For  $HG$  (d) touches  
the circle  $DEF$ ; (e) to which  $IK$  is parallel, and placed  
outwardly; (f) wherefore  $IK$  does not touch the cir-  
cle  $DEF$ ; much less do  $CI, CK$ , and the other sides  
of the polygon more remote from the center. Which  
was to be done.

a 30. 3.  
b 1. 10.

c sch. 16. 4.  
d cor. 16. 3.  
e 28. 1.  
f 34. def. 1.

Coroll.

Observe that  $IK$  touches not the circle  $DEF$ .

PROP. XVII. Fig. 36.

Two spheres  $ABCV, EFGH$ ; consisting about the same  
center  $D$ , being given, to inscribe a solid of many sides (or  
polyhedron) in the greater sphere  $ABCV$ , which shall not  
touch the superficies of the lesser sphere  $EFGH$ .

Let both the spheres be cut by a plane passing thro'  
the center, making the circles  $EFGH, ABCV$ ; and the  
diameters  $AC, BV$  drawn; cutting perpendicularly. In  
the circle  $ABCV$ , (a) inscribe the equilateral poly-  
gon  $VMLNC$ , &c. not touching the circle  $EFGH$ : Then  
draw the diameters  $Na$ ; and erect  $DO$  perpendicular  
to the plane  $ABC$  thro'  $DO$ ; and thro' the diameters  
 $Na$  conceive planes  $DOC, DON$  erected, which  
shall be (b) perpendicular to the circle  $ABCV$ ; and so in  
the

a 16. 12.  
b 18. 11.

c cor. 33. 6. the superficies of the sphere make (c) the quadrants DOC, DON. In which let the right lines CP, PQ, QR, RO, NS, ST, T<sub>2</sub>, & O (d) be fitted, equal, and of equal multitude with CN, NL, &c. make the same construction in the other quadrants OL, OM, &c. and in the whole sphere. Then I say the thing required is done.

From the points P, S, to the plane ABCV draw the perpendiculars PX, SY, (e) which shall fall on the sections AC, N a. Therefore because both (f) the right-angles PXC, SYN, (g) and PCX, SNY insisting on (h) equal circumferences, (f) are equal, the triangles also PCX, SNY (h) are equiangular. Wherefore since PC (k) = SN (l) also is PX = SY, (l) and XC = YN; (m) whence DX = DY, (n) and therefore DX : XC :: DY : YN; (o) therefore YX, NC are parallels, but because PX, SY are equal, and since being perpendicular to the same plane ABCV, they are also (p) parallels, (q) therefore YX, SP shall be equal and parallel, (r) whence SP, NC, are parallel one to the other; and so the (s) quadrilateral NCPS, and for the same reason SPQT, TQRG, as also the (t) triangle & RO are so many planes. In like manner the whole sphere may be shewn full of such quadrilaterals and triangles, wherefore the figure inscribed is a polyhedron.

From the center D (u) draw DZ perpendicular to the plane NCPS; and join ZN, ZC, ZS, ZP. Because DZ : NC (x) :: DY : YX, thence NC is (y)  $\perp$  YX (SP.) and in like manner SP  $\perp$  TQ, and TQ  $\perp$  & R. And because the angles DZC, DZN, DZS, DZP (z) are right, and the sides DC, DN, DS, DP, (a) equal, and DZ common (b) thence ZC, ZN, ZS, ZP are equal one to the other and consequently about the quadrilateral NCPS, (c) a circle may be described, in which (because NS, NC, CP are (d) equal, and NC  $\perp$  SP) NC (e) subtends more than a quadrant, (f) therefore the angle NZC at the center is obtuse, (g) therefore NCq  $\perp$  2 ZCq (ZCq + ZNq). Let NI be drawn perpendicular to AC, therefore since the angle ADN (h) DNC -  $\perp$  DCN (k) is obtuse, the angle of it DCN shall be greater than the half of a right angle; and so that which remains of the right-angle CNI shall be less than it, (n) whence IN  $\perp$  IC, therefore NCq (NIq -  $\perp$  ICq) (o)  $\perp$  2 INq. therefore IN  $\perp$  NC and consequently DZ (p)  $\perp$  DI; but the point I is (q) without the sphere EFGH, and so, much more, the point D wherefore the plane NCPS, (of which (r) the nearest point to the center is Z.) does not touch the sphere, EFGH.

Spheres  
meters BC  
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And if a perpendicular  $DS$  be drawn to the plane  $SPQT$ , the point  $S$ , and so also the plane  $SPQT$  is yet further removed from the center, which is also true of the other planes of the polyedron. Therefore the polyedron  $ORQPCN$ , &c. inscribed in the greater sphere, does not touch the lesser. *Which was to be done.*

Coroll.

Hence it follows, that if in any other sphere a solid polyedron, like to the abovesaid solid polyedron, be inscribed, the proportion of the polyedron in one sphere to the polyedron in the other is triplicate of that of the diameters of the Spheres.

For if right-lines be drawn from the centers of the spheres to all the angles of the bases of the said polyedrons, then the polyedrons will be divided into pyrs: equal in number and like; whose homo. sides are semidiameters of the spheres; as appears, if the lesser of these spheres be conceived described within the greater about the same center. For the right-lines drawn from the center of the sphere to the angles of the bases will agree one to the other by reason of the likeness of the bases; and so will like pyramids be made. Wherefore since every pyr. in one sphere to every pyr. like it in the other sphere (a) has proportion triplicate to that of the homologous sides, that is, of the semidiameter of the spheres; and (b) as one pyr. is to one pyr. so all the pyrs. that is, the solid polyedron composed of these, are to all the pyrs. that is, the solid polyedron composed of the others; therefore the polyedron of one sphere shall have to the polyedron of the other sphere, proportion triplicate to that of the semidiameters, (c) and so of the diameters of the spheres.

a cor. 8. 12.

b 12. 5.

c 15. 5.

P R O P. XVIII. Plate VI. Fig. 38.

Spheres  $BAC$ ,  $EDF$ , are in triplicate ratio of their diameters  $BC$ ,  $EF$ .

Let the sphere  $BAC$  be to the sphere  $G$  in tripli. proportion of that of the diameter  $BC$  to the diameter  $EF$ . I say  $G = EDF$ . For if it be possible, let  $G$  be  $\neq EDF$ . and conceive the sphere  $G$  concentrical with  $EDF$ . In the sphere  $EDF$  (a) inscribe a polyedron not touching the sphere  $G$ , and a like polyedron in the sphere  $BAC$ . These polyedrons (b) are in triplicate proportion of the

a 17. 12.

bcor. 17. 12.

c hyp.  
d 14. 5.

diameters BC, EF, (c) that is, of the sphere BAC to G. (d) consequently the sphere G is greater than the polyedron inscribed in the sphere EDF, the part than the whole.

e hyp. inverse.  
f 14. 5.

Again, if it be possible, let the sphere G be  $\sqsubset$  EDF. and as the sphere EDF is to another sphere H, so let G be to BAC, (e) that is, in triplicate proportion of the diameter EF to BC, therefore since BAC (f)  $\sqsubset$  H, we shall incur the absurdity of the first part, wherefore rather the sphere G = EDF. Which was to be demonstrated.

Coroll.

Hence, as one sphere is to another sphere, so is a polyedron described in that to a like polyedron described in this..

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The End of the Twelfth Book.

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# The THIRTEENTH BOOK

## O F

# E U C L I D E's

## E L E M E N T S.

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### PROPOSITION I. Plate VI. Fig. 39.

**I**F a right-line  $z$  be divided according to extreme and mean proportion ( $z : a :: a : e$ .) the square of the half of the whole line  $z$ , and of the greater segment  $a$ , as one line, is quintuple to that which is described of half of that whole line  $z$ .

I say  $Q. a + \frac{1}{2}z = 5 Q. \frac{1}{2}z$ . (a) that is  $aa + \frac{1}{4}zz + za = zz + \frac{1}{4}zz$  (b) or  $aa + za = zz$ . For  $ze + za$  (c)  $= zz$ . and  $ze$  (d)  $= aa$ . (e) therefore  $aa + za = zz$ . Which was to be demonstrated.

a 4. 2.  
b 3. ax. 1.  
c 2. 2.  
d hyp. and  
16. 6.  
e 2. ax. 3.  
1. ax.

### PROP. II. Fig. 39.

If a right-line  $\frac{1}{2}z + a$  be in power quintuple to a segment of it self  $\frac{1}{2}z$ . the line double of the said segment ( $z$ ) being divided according to extreme and mean proportion, the greater segment is (a) the other part of the right-line at first given  $\frac{1}{2}z + a$ .

I say  $z : a :: a : e$ . Because by the hyp. \*  $aa + \frac{1}{4}zz + za = zz + \frac{1}{4}zz$ ; or  $aa + za = zz$  (a)  $= ze + za$ , (b) thence shall  $aa = ze$ . (c) wherefore  $z : a :: a : e$ . Which was to be demonstrated.

\* 4. 2.  
a 2. 2.  
b 3. ax.  
c 1. 6.



## PROP. III. Plate VI. Fig 40.

If a right-line  $z$  be divided according to extreme and mean proportion ( $z : a :: a : e$ ) the line made of the less segment  $e$  and half of the greater segment  $a$ , is in power quintuple to the square, which is described of the half line of the greatest segment  $a$ .

- a 4. 2.  
b 3. ax.  
c 3. 2.  
d hyp. and  
17. 6.

I say  $Q : e + \frac{1}{2}a = 5 Q : \frac{1}{2}a : (a)$  that is  $ee + \frac{1}{4}aa + ea = aa + \frac{1}{4}aa$ . (b) or  $ee + ea = aa$ . For  $ee + ea (c) = ze (d) = aa$ . Which was to be demonstrated.

## PROP. IV. Fig. 41.

If a right-line  $z$  be cut according to extreme and mean proportion ( $z : a :: a : e$ ) the square made of the whole line  $z$ , and that made of the lesser segment  $e$ , both together, are triple of the square made of the greater segment  $a$ .

- a 4. 1.  
b 3. 2.  
c 17. 6.  
d 2. ax.

I say  $zz + ee = 3 aa$ . (a) or  $aa + ee + 2ae + ee = 3aa$ . For  $ae + ee (b) = ze (c) = aa$ . (d) therefore  $aa + 2ae + 2ee = 3aa$ . Which was to be demonstrated.

## PROP. V.

D——A——C——B

If a right-line  $AB$  be cut according to extreme and mean proportion in  $C$ , and a line  $AD$ , equal to the greater segment  $BC$ , added to it, the whole right-line  $DB$  is divided according to extreme and mean proportion; and the greater segment is the right-line  $AB$  given at the beginning.

- a hyp.

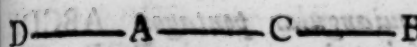
For because  $AB : AD. (a) :: AC : CB$ . and by inversion  $AD : AB :: CB : AC$ . therefore by composition  $DB : AB :: AB : AC (AD)$ . Which was to be demonstrated.

Schol.

But if  $BD : BA :: BA : AD$ . then shall be  $BA : AD :: AD : BA - AD$ . For by dividing  $BD - BA (AD) : BA :: BA - AD : AD$ . therefore inversely  $BA : AD :: AD : BA - AD$ .

## PROP.

PROP. VI.

 If a rational right-line AB be cut according to extreme and mean proportion in C, either of the segments (AC, CB) is an irrational-line of that kind which is called apotome or residual.

To the greater segment AC (a) add  $AD = \frac{1}{2} AB$ . (b) therefore  $DCq = 5 DAq$ . (c) therefore  $DCq \supset DAq$ . consequently (d) since AB, (e) and so the half thereof DA are  $\rho$ , likewise DC is  $\rho$ . But because  $5 : 1 ::$  not  $Q : Q$ . (f) thence is  $DC \supset DA$ . (g) therefore  $DC - AD$ , that is, AC, is a residual-line. Further, because  $ACq (b) = AB \times BC$ , and AB is  $\rho$ , (i) likewise BC is a residual-line. Which was to be demonstrated.

a 3. 1.  
b 1. 13.  
c 6. 10.  
d hyp.  
e sch. 12.  
10.  
f 9. 10.  
g 74. 10.  
h 17. 6.  
i 98. 10.

PROP. VII. Plate VI. Fig. 42.

If three angles of an equilateral pentagon ABCDE, whether they follow in order, (EAB, ABC, BCD,) or not, (EAB, BCD, CDE) are equal, the pentagon ABCDE shall be equiangular.

Let the right-lines BE, AC, BD, be subtended to the equal angles in order.

Because the sides EA, AB, BC, CD, and the included angles (a) are equal, (b) therefore shall the bases BE, AC, BD, (c) and the angles AEB, ABE, BAC, BCA, be equal. (d) Wherefore  $BF = FA$ , (e) and consequently  $FC = FE$ ; therefore the triangles FCD, FED, are equilateral one to the other: (f) whence the angle  $FCD = FED$ . (g) consequently the angle  $AED = BCD$ . In like manner the angle CDE is equal to the rest; wherefore the pentagon is equiangular. Which was to be demonstrated.

But if the angles EAB, BCD, CDE, which are not in order, be supposed equal, (b) then shall the angle  $AEB = BDC$ , and  $BE = BD$ . (k) and thence the angle  $BED = BDE$ . (l) consequently the whole angle  $AED = CDE$ , therefore because the angles A, E, D, in order, are equal, as before, the pentagon shall be equiangular. Which was to be demonstrated.

a hyp.  
b 4. 1.  
c 4. and 5.  
1.  
d 6. 1.  
e 3. ax. 1.  
f 8. 1.  
g 2. ax. 1.  
h 4. 1.  
k 5. 1.  
l 2. ax.

## P R O P. VIII. Plate VI. Fig. 43.

If in an equilateral and equiangular pentagon  $ABCDE$ , two right-lines  $BD$ ,  $CE$ , subtend two angles  $BCD$ ,  $CDE$  following in order, those lines cut one another according to extreme and mean proportion; and their greater segments  $BF$  or  $EF$  are equal to the side of the pentagon  $BC$ .

a 14. 4.  
b 28. 3.  
c 27. 3.  
d 32. 1.  
e 33. 6.  
f 6. 1.  
g 27. 3.  
h 4. 6.

(a) Describe about the pentagon the circle  $ABD$ . (b) The arch  $ED$  is  $= BC$ , (c) therefore the angle  $FCD = FDC$ . (d) therefore the angle  $BFC = 2 FCD$  ( $FCD + FDC$ .) But the arch  $BAE$  is  $= 2 ED$ , and consequently the angle  $BCF$  (e)  $= 2 FCD = BFC$ . (f) wherefore  $BF = BC$ . Which was to be demonstrated. Moreover, because the triangles  $BCD$ ,  $FCD$ , are (g) equiangular. (h) therefore  $BD : DC$  ( $BF$ ) : :  $CD : (BF) FD$ . and likewise  $EC : EF$  : :  $EF : FC$ . Which was to be demonstrated.

## P R O P. IX. Fig. 44.

If the side of an Hexagon  $BE$ , and the side of a Decagon  $AB$  both described in the same circle  $ABC$ , be added together, the whole right-line  $AE$  is cut according to extreme and mean proportion ( $AE : BE$  : :  $BE : AB$ .) and the greater segment therefore is the side of the Hexagon  $BE$ .

a hyp. and  
27. 3.  
b 32. 1.  
c 7. ax. 1.  
d 5. 1.  
e 1. ax. 1.  
f 4. 6.  
g cor. 15. 4.

Draw the diameter  $ADC$ , and join the right-lines  $DB$ ,  $DE$ . Because the angle  $BDC$  (a)  $= 4 BDA$  and the angle  $BDC$  (b)  $= 2 DBA$  ( $DAB + DBA$ ) thence shall  $DBA$  (b) ( $DBE + BED$ ) (c) be  $= 2 BDA$  (d)  $= 2 BDE$ , whence the angle  $DBA$  or  $DAB$  (e)  $= ADE$ . Therefore the triangles  $ADE$ ,  $ADB$ , are equiangular: (f) wherefore  $AE : AD$  (g) ( $BE$ ) : :  $AD : (BE) AB$ . Which was to be demonstrated.

Coroll.

Hence, if the side of a hexagon in a circle be cut according to extreme and mean proportion; the greater segment thereof shall be the side of the decagon in the same circle.

P R O P.



PROP. X. Plate VII. Fig. 1.

If an equilateral Pentagon  $ABCDE$  be inscribed in a circle  $ABCE$ , the side of the pentagon  $AB$  containeth in power both the side of a hexagon  $FB$ , and the side of a decagon  $AH$  inscribed in the same circle.

Draw the diameter  $AG$ , and bisect the arch  $AH$  in  $K$ , and draw  $FK$ ,  $FH$ ,  $FB$ ,  $BH$ ,  $HM$ .

The semicircle  $AG$  — the arch  $AC$  ( $a$ ) =  $AG$  —  $AD$ . that is, the arch  $CG$  =  $GD$  ( $b$ ) =  $AH$  =  $HB$ . therefore the arch  $BCG$  =  $2$   $BHK$ ; ( $c$ ) and so the angle  $BFG$  =  $2$   $BFK$ . ( $d$ ) but the angle  $BFG$  =  $2$   $BAG$ ; ( $e$ ) therefore the angle  $BFK$  =  $BAG$ . Wherefore the triangles  $BFM$ ,  $FAB$ , ( $f$ ) are equiangular; ( $g$ ) whence  $AB : BF :: BF : BM$ ; ( $h$ ) therefore  $AB \times BM = BF^2$ . Moreover the angle  $AFK$  ( $k$ ) =  $HFK$ , and  $FA = FH$ ; ( $m$ ) wherefore  $AL = LH$ , ( $m$ ) and the angles  $FLA$ ,  $FLH$  are equal, and so right-angles, therefore the angle  $LHM$  ( $n$ ) =  $LAM$  ( $n$ ) =  $HBA$ ; therefore the triangles  $AHB$ ,  $AMH$ , ( $o$ ) are equiangular; wherefore  $AB : AH :: AH : AM$ ; ( $p$ ) therefore  $AB \times AM = AH^2$ . Since therefore  $ABq$  ( $r$ ) =  $AB \times BM + AB \times AM$ , ( $s$ ) thence  $ABq = BF^2 + AH^2$ . Which was to be demonstrated.

Coroll.

1. Hence, a right-line ( $FK$ ) which being drawn from the center ( $F$ ) divides an arch ( $HA$ ) into two equal segments, does also divide the right-line ( $HA$ ) subtending that arch perpendicularly into two equal segments.

2. The diameter of a circle ( $AG$ .) drawn from any angle ( $A$ ) of a pentagon, does divide equally in two, both the arch ( $CD$ .) which the side of the pentagon opposite to that angle subtends, and also the opposite side it self ( $CD$ ) and that perpendicularly.

Schol. Fig. 2.

Here, according to our promise, we shall lay down a ready praxis of the 11th prop. of the 4th Book.

Problem

To find the side of a pentagon to be inscribed in a circle  $ADB$ .

Draw

a 28. 3. 6.  
3 ax.  
b hyp. and  
7. ax.  
c 33. 6.  
d 20. 3.  
e 1. ax. 1.  
f 32. 1.  
g 4. 6.  
h 17. 6.  
k 27. 3.  
m 4. 1.  
n 27. 3.  
o 32. 1.  
p 4. 6.  
q 17. 6.  
r 2. 2.  
s 2. ax.

Draw the diameter AB, to which erect a perpendicular CD at the center C, divide CB equally in E, and make  $EF = ED$ , then DF shall be the side of the pentagon.

For  $BF \times FC + ECq$  (a)  $= EFq$  (b)  $EDq$  (c)  $= DCq + ECq$ . (d) therefore  $BF \times FC = DCq$  or  $ECq$ . (e) wherefore  $BF : EC :: BC : FC$ ; therefore since BC is the side of a hexagon, (f) FC shall be the side of a decagon. Consequently  $DF$  (b)  $= \sqrt{DCq + FCq}$  (g) is the side of a pentagon. Which was to be done.

a 6. 2.  
b *confr.*  
c 47. 1.  
d 3. *ax.*  
e 17. 6.  
f 9. 13.  
g 10. 13.  
h 47. 1.

PROP. XI. Plate VII. Fig. 3.

If in a circle ABCD, whose diameter AG, is rational, an equilateral pentagon be inscribed ABCDE; the side of the pentagon AB is an irrational-line of that kind which is called a minor-line.

Draw the diameter BFH, and the right-lines AC, AH; and \* make  $FL = \frac{1}{4}$  of the radius FH; and  $CM = \frac{1}{2} CA$ .

Because the angles AKF, AIC, are (a) right-angles, and CAI common, the triangles AKF, AIC, are (b) equiangular: (c) therefore  $CI : FK$  (c)  $:: CA : FA$  (FB) (d)  $:: CM : FL$ ; therefore by permutation  $FK : FL :: CI : CM$  (d)  $:: CD : CK$  (2 CM) and so by (e) composition  $CD + CK : CK :: KL : FL$ ; (f) consequently  $Q : CD + CK$  (g)  $(5 CKq) : CKq :: KLq : FLq$ . therefore  $KLq = 5 FLq$ . wherefore if BH (p) be taken 8, FH shall be 4. FL 1, and FLq 1, BL 5, and BLq 25, KLq 5. by which it appears that BL and KL are p (h)  $\sqrt{25}$ , (k) and so BK is a residual and KL its congruent or adjoining-line; but since  $BLq - KLq = 20$ . (l) thence  $BL = \sqrt{BLq - KLq}$  \* whence BK shall be a fourth residual-line. Therefore because  $ABq$  (m) is  $= HB \times BK$ , (n) shall AB be a minor-line. Which was to be demonstrated.

\* 10. 6.  
a *cor.* 10.  
13.  
b 32. 1.  
c 4. 6.  
d 15. 5.  
e 18. 5.  
f 22. 6.  
g 1. 13.  
h 9. 10.  
k 74. 10.  
l 9. 10.  
\* 4 *def.* 85.  
10.  
m *cor.* 8. 6.  
and 17. 6.  
n 95. 10.

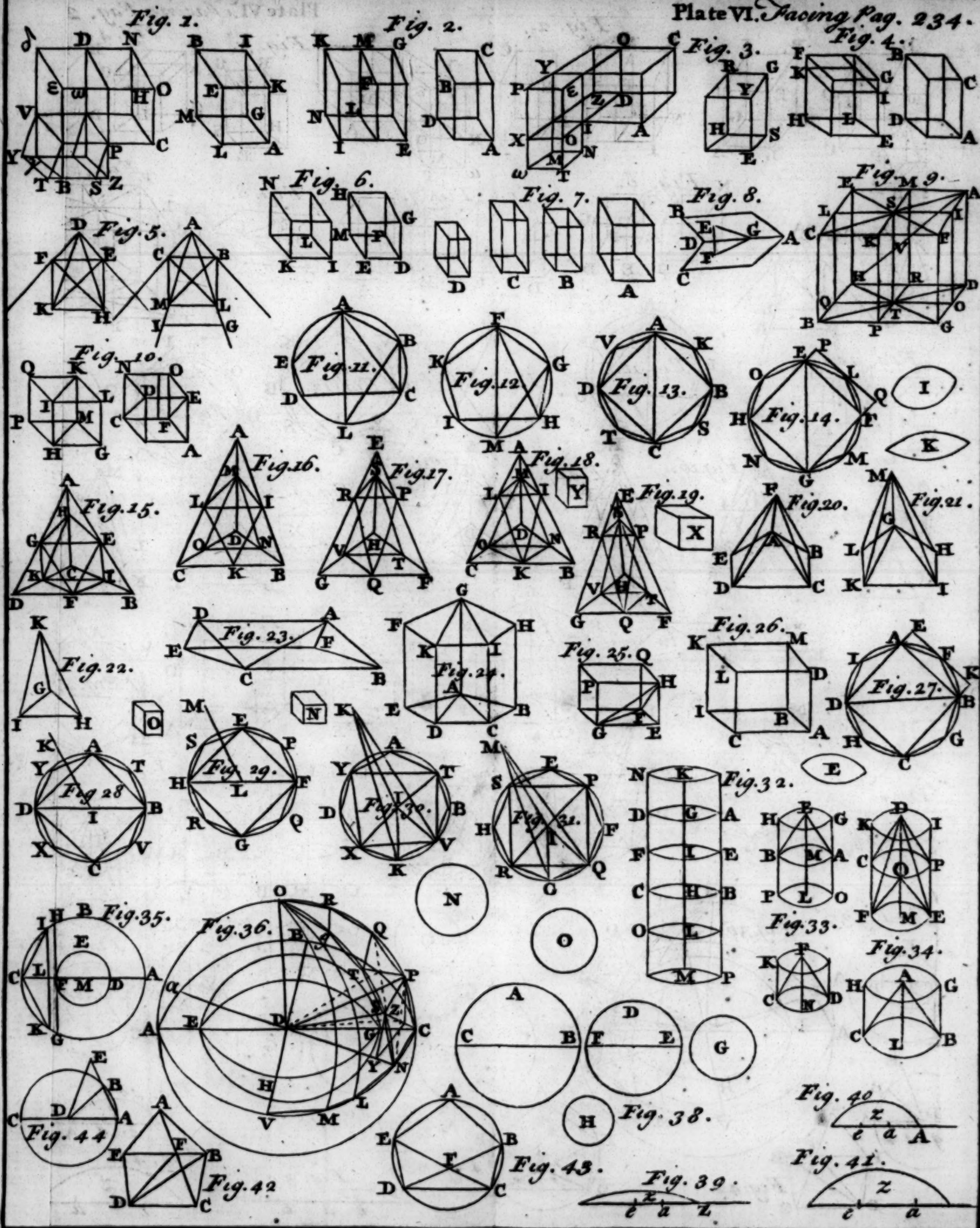
PROP. XII. Fig. 4.

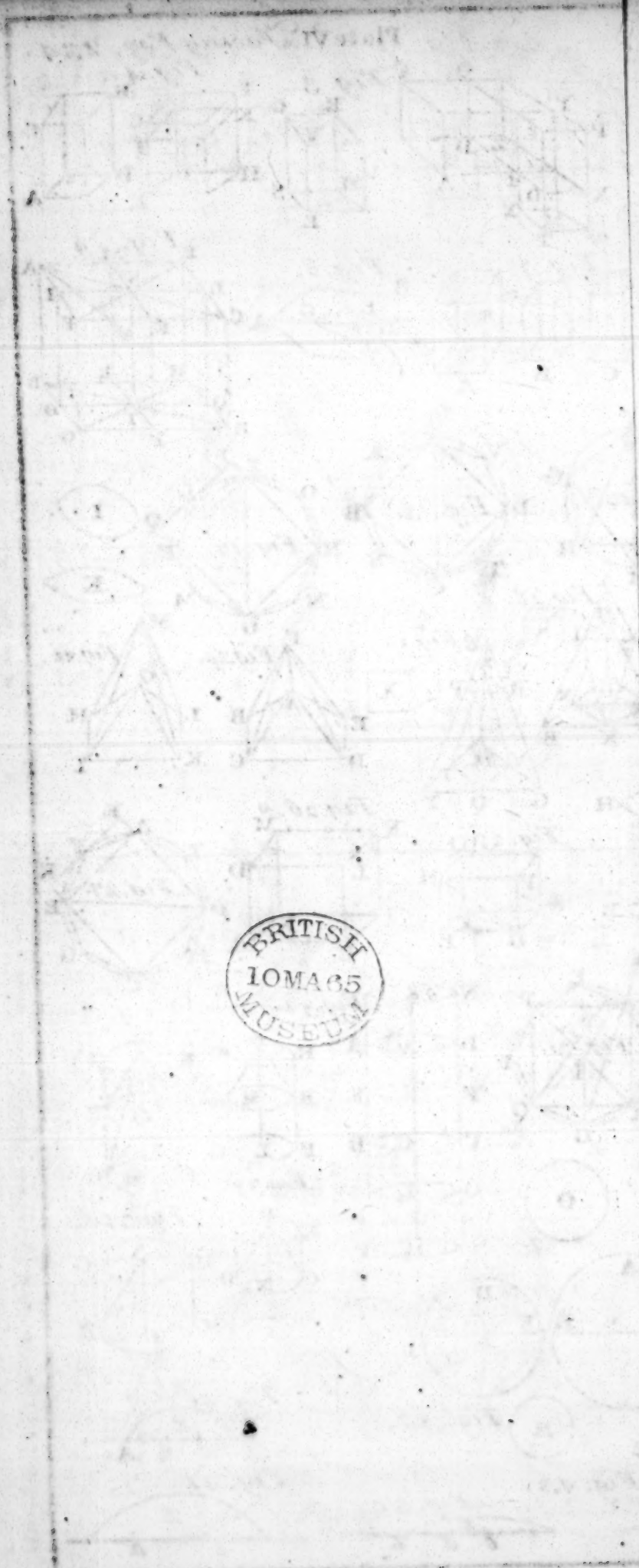
If in a circle ABEC an equilateral triangle ABC be inscribed, the side of that triangle AB is in power triple to the line AD drawn from D the center of the circle to the circumference,

The

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shall b



The diameter being extended to E, draw BE. Be-  
cause the arch BE (a) = EC, the arch BE is the sixth  
part of the circumference, (b) therefore BE = DE.  
Hence AEq (c) = 4 DEq (4 BEq) (d) = ABq + BEq  
(ADq.) (e) consequently ABq = 3 ADq. Which was  
to be dem.

a cor. 10.  
13.  
b cor. 15. 4.  
c 4. 2.  
d 47. 1.  
e 3. ax. 1.

Coroll.

1. AEq : ABq : : 4 : 3.
2. ABq : AFq : : 4 : 3. (f) For ABq : AFq : : AEq :
3. DF = FE. For the triangle EBD (g) is equi-  
lateral, (h) and BF perpendicular to ED : (b) there-  
fore EF = FD.
4. Hence, AF = DE + DF = 3 DF.

f cor. 8. 6.  
and 22. 6.  
g cor 15. 4.  
h cor. 3. 3.

P R O P. XIII. Plate VII. Fig. 5, 6.

To describe a pyramid EGFI, and comprehend it in a  
sphere given : and to demonstrate that the diameters of the  
sphere AB is in power sesquialter of the side EF of the  
pyramid EGFI.

About AB describe the semicircle ADB ; (a) and let  
AC = 2 CB. From the point C erect the perpendi-  
cular CD, and join AD, DB, then at the interval of the  
radius HE = CD describe the circle HEFG. (b) wherein  
describe the equilateral triangle EFG ; from H (c) erect IH  
perpendicular to the plane EFG, produce IH to  
(d) so that IK = AB ; and join the right-lines IE, IF,  
Then EFGI shall be the pyramid required.

a 10. 6.  
b cor. 15. 4.  
c 12. 11.  
d 3. 1.  
e constr.  
f 4. 1.  
g 20. 6.  
h 2. ax.  
k 12. 13.  
l 1. ax. 1.

For because the angles ACD, IHE, IHF, IHG, (e) are  
right-angles ; and CD, HE, HF, HG (e) equal, (e) and  
AC = AC ; (f) therefore AD, IE, IF, IG, shall be equal  
among themselves. But because AC (2 CB,) : CB (g) : :  
ACq : CDq. thence shall ACq = 2 CDq ; therefore ADq  
= ACq + CDq (h) = 3 CDq = 3 HEq (k) = FFq ;  
therefore AD, EF, IE, IF, IG are equal, and so the  
pyramid EFGI is equilateral. But if the point C be placed  
on H, and AC upon HI, the right-lines AB, IK, (m)  
shall agree, as being equal. Wherefore the semicircle  
ADB being drawn about the axis AB or IK (n)  
shall pass by the points E, F, G, \* and so the pyramid EF-  
GI shall be inscribed in a sphere. Which was to be done.

m 8. ax. 1.  
n 15. def. 1  
\* 31. def.  
11.

Also



o cor. 8. 6.  
p constr.

Also it is manifest that  $BAq : ADq(o) :: BA : AC$   
 $:: 3 : 2$ . Which was to be demonstrated.

Coroll.

q 12. 13.

1.  $ABq : HEq :: 9 : 2$ . For if  $ABq$  be put = then  $ADq$  ( $EFq$ ) shall be 9. ( $q$ ) consequently  $HEq$  shall be 2.

r constr.

2. If  $L$  be the center, then shall  $AB : LC :: 6 : 1$ .  
1. For if  $AB$  be put = 6. then  $AL$  shall be 3; and thence  $AC$  4. wherefore  $LC$  shall be 1. Hence  
3.  $AB : HI :: 6 : 4 :: 3 : 2$ . whence.  
4.  $ABq : HIq :: 9 : 4$ .

P R O P. XIV. Plate VII. Fig. 7, 8.

To describe an Octaedron KEFGDL, and comprehend it in the given sphere, wherein a pyramid is: and to monstrate that  $AH$ , the diameter of the sphere, is power double of  $AC$ , the side of that Octaedron.

a 46. 1.

b 12. 11.

c 3. 1.

About  $AH$  describe the semicircle  $ACH$ , and from the center  $B$  erect the perpendicular  $BC$ , draw  $AC$ ,  $HC$  then upon  $ED = AC$  ( $a$ ) make the square  $EFGD$ , whose diameters  $DF$ ,  $EG$ , cut in the center  $I$ ; from  $I$ , draw  $IL = AB$  ( $b$ ) perpendicular to the plane  $EFGD$ , produce  $IL$  ( $c$ ) till  $IK = IL$ . and join  $KE$ ,  $KF$ ,  $KG$ ,  $KD$ ,  $LE$ ,  $LF$ ,  $LG$ ,  $LD$ ; then shall  $KEFGDL$  be the Octaedron required.

d 4. 1.

For  $AB$ ,  $BH$ ,  $FI$ ,  $IE$ , &c. being semidiameters equal squares are equal one to the other; ( $d$ ) whence bases  $LF$ ,  $LE$ ,  $FE$ , &c. of the right-angled triangles  $LIE$ ,  $LIF$ ,  $FIE$ , &c. are equal, and consequently the eight triangles  $LFE$ ,  $LFG$ ,  $LGD$ ,  $LDE$ ,  $KEF$ ,  $KFG$ ,  $KGD$ ,  $KDE$ , are equilateral, ( $e$ ) and make an Octaedron, which may be inscribed in a sphere, whose center is  $I$ , and  $IL$  or  $AB$  the radius, (because  $AB$ ,  $IL$ ,  $IF$ ,  $IK$ , &c. ( $f$ ) are equal.) Which was to be done. Moreover, is evident that  $AHq$  ( $LKq$ ) ( $g$ ) = 2  $ACq$  (2  $LD$ ) Which was to be demon.

e 27. def.  
11.

f constr.

g 47. 1.

Coroll.

1. Hence it is manifest, that in the Octaedron three diameters  $EG$ ,  $FD$ ,  $LK$  cut one the other perpendicularly in the center of the sphere.

†

2. A

2. Also, that the three planes EFGD, LEKG, LFKD  
squares, cutting one another perpendicularly.

3. The Octaedron is divided into two like and equal  
pyramids EFGDL, and EFGDK, whose common base  
the square EFGD.

4. Lastly, it follows (b) that the opposite bases of the  
Octaedron are parallel one to the other.

h 15. 11.

P R O P. XV. Plate VII. Fig. 9, 10.

To describe a cube EFGHIKLM, and comprehend it  
in the same sphere, wherein the former figures were; and  
demonstrate that AB the diameter of the sphere is in  
power triple to EF, the side of that cube.

Upon AB describe a semicircle ACB; (a) and make  
 $AB = 3 DA$ , and from D raise the perpendicular DC,  
and join BC and AC. Then upon  $EF = AC$  (b) make  
the square EFGH, upon whose plane let the right-lines  
EI, FK, HM, GL, stand perpendicular, being equal to  
EF, and connect them with the right-lines IK, KL, LM,  
IM. The solid EFGHIKLM, is a cube, as is suffici-  
ently apparent from the construction.

a 10. 6.

b 46. 1.

In the opposite squares EFKI, HGML, draw the di-  
ameters EK, FI, HL, MG, through which let the planes  
EKLH, FIMG be drawn, cutting one another in the line  
NO, which (c) shall divide equally in two parts the diame-  
ters of the cube EL, FM, GI, HK, in P the center of the  
cube; (d) therefore P shall be the center of a sphere passing  
through the angular points of the cube. Moreover,  
 $ELq (e) = EKq + KLq (e) = 3 KLq, (f) \text{ or } 3 ACq.$   
but  $ABq : ACq (g) :: BA : DA (f) :: 3 : 1$ ; (b) there-  
fore  $AB = EL$ ; wherefore we have made a cube, &c.  
Which was to be done.

c cor. 39.

11.

d 15. def.

1. and 14.

def. 11.

e 47. 1.

f constr.

g cor. 8. 6.

h 14. 5.

Coroll.

1. Hence it is manifest that all the diameters of the  
cube are equal one to the other, and do equally bisect one  
another in the center of the sphere. And for the same  
reason the right-lines which conjoin the center of the  
opposite squares are bisected in the same center.

2. The diameter of a sphere containeth in power the  
side of a tetraedron and of a cube, viz.  $ABq (k) = (l)$   
 $BCq + (m) ACq.$

k 47. 1.

l 13. 13.

m 15. 13.

P R O P.

## P R O P. XVI. Plate VII. Fig. 11, 12.

To describe an Icosaedron ZGHKIFYVXRST, and encompass it in the sphere, wherein were contained the foresaid solids; and to demonstrate that FG the side of the Icosaedron is that irrational-line, which is called minor-line.

a 10. 6.

b 11. 4.

c 12. 11.

d constr.

e 6. 11.

f 33. 1.

g 15. 11.

h 1. def. 3.

k 47. 1.

l constr.

m 10. 12.

n sch. 48. 1.

and ax. 1.

o cor. 14.

11.

p 47. 1.

q 10. 13.

Upon AB the diameter of a sphere describe the semicircle ADB; and (a) make AB equal 5 BC, then from C erect CD perpendicular, and draw AD and BD. At the distance EF = BD describe the circle EFKNG. (b) wherein inscribe the equilateral pentagon FKIHO. Divide equally in two parts the arches FG, GH, &c. and join the right-lines FL, LG, &c. being the sides of a decagon. Then (c) erect EQ, LR, MS, NT, OV, PX equal to EF, and perpendicular to the plane FKNQ; and connect RS, ST, TV, VX, XR; as also FX, FR, GR, GS, HS, HT, IT, IV, KV, KX. Lastly, produce EQ, and take QY = FL, and EZ = FL, and conceive the right-lines ZG, ZH, ZI, ZK, ZF to be drawn; as also YV, YX, YR, YS, YT. Then I say the Icosaedron required is made.

For because EQ, LR, MS, NT, OV, PX, are (a) equal and (e) parallel, also those lines that join them EL, QR, EM, QS, EN, QT, EO, QV, EP, QX (f) are equal and parallel. And thence likewise LM (or FG) RS, MN, ST, &c. are equal one to the other; (g) therefore the plane drawn through EL, EM, &c. is equidistant from the plane passing through QR, QS, &c. (b) and the circle QXRSTV drawn from the center Q is equal to the circle EPLMNO, and RSTVX is an equilateral pentagon. But EF, EG, EH, &c. and QX, QR, QS, &c. being conceived to be drawn; then because FRq (k) = FLq + LRq, (l) or EFq (m) = FGq. (n) therefore FR, FG, and so all RS, FG, FR, RG, GS, GH, &c. shall be equal one to the other, and consequently the ten triangles RFX, KFG, RGS, &c. are equilateral and equal. Moreover, because XQY is a (o) right-angle; therefore XYq (p) = QXq + QYq (q) = VXq or FGq; wherefore XY, VX, and likewise YV, YT, YS, YR, ZG, ZH, &c. are equal. Therefore other ten triangles are made, equilateral and equal both to one another, and to the ten former, and so an Icosaedron is made.

Moreover



Moreover, divide equally EQ in  $a$ , draw the right lines,  $aF$ ,  $aX$ ,  $aV$ ; and because QX ( $r$ ) = QV, and  $aQ$  the common side, and EQX, EQV are right-angles ( $f$ ) therefore shall  $aX$  be =  $aV$ ; and for the same reason all the lines  $aX$ ,  $aR$ ,  $aS$ ,  $aT$ ,  $aV$ ,  $aF$ ,  $aG$ ,  $aH$ ,  $aI$ ,  $aK$  are equal. But because ZQ:QE ( $t$ ): : QE:ZE, therefore Zaq ( $u$ ) = 5 Eaq ( $x$ ) = EQq (EFq) + Eaq ( $y$ ) = aFq; therefore Za = aF; in like manner aF = Ya; therefore the sphere, whose center is  $a$  and aF the radius, shall pass through the 12 angular points of the Icosaedron.

Lastly, ( $x$ ) because Za : aE : : ZY : QE; ( $a$ ) and so Zaq : aEq : : ZYq : QEq. ( $b$ ) therefore ZYq = 5 QEq, or 5 BDq; but ABq : BDq ( $c$ ): : AB : BC : : 5 : 1; ( $d$ ) therefore ZY = AB, Which was to be done. Therefore if AB be put  $\rho$ , ( $e$ ) then EF =  $\sqrt{AB \times BC}$  shall be also  $\rho$ , and consequently FG the side of the pentagon, and likewise of the Icosaedron, ( $f$ ) is a minor-line. Which was to be dem.

Coroll.

1. From hence is inferred, that the diameter of the sphere is in power quintuple of the semidiameter of the circle encompassing the five sides of the Icosaedron.

2. Also it is manifest that the diameter of the sphere is composed of the side of a hexagon, that is, of the semidiameter, and two sides of the decagon of a circle encompassing the five sides of the Icosaedron.

3. It appears likewise that the opposite sides of an Icosaedron, such as RX, HI, are parallels. For RX ( $a$ ) is parallel to LP, ( $b$ ) parallel to HI.

P R O P. XVII. Plate VII. Fig. 13.

To describe a Dodecaedron, and comprehend it in the sphere wherein the former figures were comprehended; and to demonstrate that the side RS of the Dodecaedron is an irrational-line of that sort which is called an apotome or residual-line.

Let AB be a cube inscribed in the given sphere, and let all the sides thereof be divided equally in the points E, H, F, G, K, L, &c. and join the right-lines KL, MH, HG, EF; ( $a$ ) make HI : IQ : : IQ : QH; and take NO, NP,

r 15. def. 1.

f 4. 1.

t 9. 13.

u 3. 13.

x 4. 2.

y 47. 1.

z 15. 5.

a 22. 6.

b 14. 5.

c cor. 8. 6.

d 1. ax. 1.

e sch. 12.

10.

f 11. 13.

a 33. 1.

b sch. 26.

3.

a 30. 6.

a 47. 1.  
 b 7. ax. 1.  
 c 4. 13.  
 d 47. 1.  
 e 4. 2.  
 f *constr.* 96.  
 11.  
 g 33. 1.  
 h 9. 1.  
 k 7. 11.  
 k *constr.*  
 l 6. 11.  
 m 32. 6.  
 n 1, and 2.  
 11.  
 o 5. 13.  
 p 47. 13.  
 q 1. ax. 6.  
 and 4. 13.  
 r 4. 2.  
 f 8. 1.  
 \* 7. 13.  
 t 15. 13.  
 u 1. ax. 1.  
 x 29. 1.  
 z 47. 1.  
 a 4. 13.  
 b 15. 13.

NP, = IQ; then erect OR, PS, perpendicular to the plane DB, and QT to the plane AC; and let OR, PS, QT, be equal to IQ, NO, NP; whence DR, RS, SC, CT, TD, being connected, DRSCCT shall be a pentagon of the Dodecaedron required. For draw NV parallel to OR, and having drawn NV out as far as the center of the cube X, join the right-lines DS, DO, DP, CR, CP, HV, HT, RX. Because DOq (a) = DKq (b) (KNq) + KOq (c) = 3 ONq (3 ORq) (d) thence DRq = 4 ORq (e) = OPq, or, RSq. therefore DR = RS. By a like way of reasoning DR, RS, SC, CT, TP are equal. But because OR (f) is equal and (g) parallel to PS, therefore RS, OP, and consequently RS, DC shall be also parallels; (h) therefore these with those that conjoin them DR, CS, VH, are in one and the same plane. Moreover, because HI : IQ (k) : : IQ (TQ) : QH (k) : : HN : NV; and both TQ, HN, and QH, NV (k) are perpendicular to the same plane, (l) and so also parallels, (m) THV shall be a right-line; (n) therefore the Trapezium DRSC, and the triangle DTS are in one plane extended through the right-lines DC, TV; (o) therefore DTCSR is a pentagon, and that also equilateral, by what is shewn already. Furthermore, because PK : KN : : KN : NP; and DSq (p) = DPq + PSq (PNq) = (p) DKq + PKq + NPq, (q) thence DSq = DKq + 3 KNq = 4 DKq (4 DHq) (r) = DCq; therefore DS = DC; whence the triangles DRS, DCT, are equilateral one to another; (s) therefore the angle DRS = DTC, and likewise the angle CSR = DCT; therefore the \* pentagon DTCSR is also equiangular. Moreover, because AX, DX, CX, &c. are semidiameters of the cube (t) thence is XN = IH, or KN, (u) and so XV = KP; wherefore because RVX, is a (x) right-angle, (z) thence RXq = XVq + RVq (NPq) = KPq + NPq (a) = 3 KNq (b) = AXq or DXq, &c. therefore RX, AX, DX, and for the same reason XS, XT, AX, are equal one to another. And if by the same method whereby the pentagon DTCSR was made, twelve like pentagons, touching the twelve sides of the cube, be made; they shall compose a Dodecaedron; and a sphere passing through their angular points, whose radius is AX, or RX, shall comprehend that Dodecaedron. Which was to be done.

Lastly, because KN : NO (c) : : NO : OK; (d) thence KL : OP : : OP : OK + PL. Therefore if AB the diameter

the sphere be supposed  $\dot{p}$ , then shall  $KL (e) = \sqrt{\frac{AB}{3}}$   
 be also  $\dot{p}$ . (g) whence OP, or RS the side of the Dodecaedron shall be a residual-line. Which was to be dem.

e 15. 3.  
 f schol. i 2.  
 10.  
 g 6. 13.

*Coroll.*

From this demonstration it follows, 1. That if the side of a cube be cut in extreme and mean proportion, the greater segment shall be the side of the Dodecaedron described in the same sphere.

2. If the lesser segment of a right-line, cut in extreme and mean proportion, be the side of the Dodecaedron, the greater segment shall be the side of the cube inscribed in the same sphere.

3. It is manifest also, that the side of the cube is equal to the right-line which subtendeth the angle of a pentagon of the Dodecaedron inscribed in the same sphere.

P R O P. XVIII. Plate VII. Fig. 14.

To find out the sides of the five precedent figures; and compare them together.

Let AB be the diameter of the sphere given, and AEB the semicircle, and let AC be (a)  $= \frac{1}{2} AB$ , and AD (b)  $= \frac{1}{3} AB$ ; then erect the perpendiculars CE, DF, and AG  $= AB$ ; join AF, AE, BE, BF, CG; and let fall the perpendicular HI from H, and CK being taken equal to CI, from K erect the perpendicular KL, and join AL. Lastly, (c) make  $AF : AO :: AO : OF$ .

Therefore 3 : 2 (d)  $:: AB : BD (e) :: ABq : BFq$  the side of a Tetraedron, and 2 : 1  $:: (a) AB : AC :: ABq : BFq (f)$  the side of an Octaedron.

Also 3 : 1 (d)  $:: AB : AD (e) :: ABq : AFq (g)$  the side of an Hexaedron.

Moreover, because  $AF : AO (b) :: AO : OF$ ; (k) hence shall AO be the side of a Dodecaedron. Lastly,  $AG (2 BC) : BC (l) :: HI : IC$ ; (m) therefore  $HI = 2 IC (n) = KI$ ; therefore  $HIq (o) = 4 CIq$ . consequently  $HIq (p) = 5 CIq$ ; (q) therefore  $ABq = 5 KIq (r)$  therefore KI, or HI is a radius of a circle enclosing the pentagon of an Icosaedron; and AK or IB, (r) is the side of a decagon inscribed in the same circle; (s) hence AL shall be the side of a pentagon, (t) and also the side of an Icosaedron. Whereby it appears that BF,

Q

BE,

a 10. 1.  
 b 10. 6.  
 c 30. 6.  
 d constr.  
 e cor. 8. 6.  
 f 14. 13.  
 g 15. 13.  
 h constr.  
 k cor. 17.  
 13.  
 l 4. 6.  
 m 14. 5.  
 n constr.  
 o 4. 2.  
 p 47. 1.  
 q 15. 5.  
 r cor. 16.  
 13.  
 s 10. 13.  
 t 16. 13.



u 1. 6.

x 4. ax. 1.

y 1. 2.

z 17. 6.

a 47. 1.

BE, AF are  $\hat{p}\hat{q}$ . and AL, AO  $\hat{p}\hat{q}$ , and BF  $\hat{p}\hat{q}$  BE, and BE  $\hat{p}\hat{q}$  AF, and AF  $\hat{p}\hat{q}$  AO. And because 3 AFq = ABq (u) = 5 KLq, and AF  $\times$  AO  $\hat{p}\hat{q}$  AF  $\times$  OF, (x) and so AF  $\times$  AO + AF  $\times$  OF  $\hat{p}\hat{q}$  2 AF  $\times$  OF, (y) that is AFq  $\hat{p}\hat{q}$  (z) 2 AOq; (a) thence shall 3 AFq (5 KLq) be  $\hat{p}\hat{q}$  6 AOq. consequently KL  $\hat{p}\hat{q}$  AO, and much rather AL  $\hat{p}\hat{q}$  AO.

That we may express these sides in numbers; If AL be supposed  $\sqrt{60}$ , then, reducing what is already shew'd to supputation, BF =  $\sqrt{40}$ , and BE =  $\sqrt{30}$ , and AL =  $\sqrt{20}$ . Also AL =  $\sqrt{30} - \sqrt{180}$  (for AK =  $\sqrt{180} - \sqrt{3}$ . and KL (HI) =  $\sqrt{12}$  :) Lastly, AO =  $\sqrt{30} - \sqrt{500} (\sqrt{25} - \sqrt{5})$

Schol.

*It is very apparent that besides the five aforesaid figures there cannot be described any other regular solid figure (viz. such as may be contained under ordinate and equal plane figures.)*

a 21. 11.

b See sch.

32. 1.

For three plane angles at least are required to the constituting of a solid angle: (a) all which must be less than four right-angles. (b) But 6 angles of an equilateral triangle, 4 of a square, and 6 of a hexagon, severally equal 4 right-angles; and 4 of a pentagon, 3 of a hexagon, 3 of an octagon, &c. exceed 4 right-angles. Therefore only of 3, 4, or 5 equilateral triangles, of squares, or 3 pentagons, it is possible to make a solid angle. Wherefore besides the five above mentioned there cannot be any other regular bodies.

### Out of P. Herigon.

*The Proportions of the sphere and the five regular figures inscribed in the same.*

Let the diameter of the sphere be 2, then shall the periphery or circumference of the greater circle be 6. 28318.

The superficies of the greater circle, 3. 14159.

The superficies of the sphere, 12. 56637.

The solidity of the sphere, 4. 1879.

The side of the tetraedron, 1. 62299.

The superficies of the tetraedron, 4. 6188.

The solidity of the tetraedron, 0. 15132.

The side of the hexaedron, 1. 1547.  
The superficies of the hexaedron, 8.  
The solidity of the hexaedron, 1. 5396.

The side of the octaedron, 1. 41421.  
The superficies of the octaedron, 6. 9282.  
The solidity of the octaedron, 1. 33333.

The side of the dodecaedron, 0. 71364.  
The superficies of the dodecaedron, 10. 51462.  
The solidity of the dodecaedron, 2. 78516.

The side of the Icosaedron, 1. 05146.  
The superficies of the Icosaedron, 9. 57454.  
The solidity of the Icosaedron, 2. 53615.

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*The End of the Thirteenth Book.*

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# The FOURTEENTH BOOK OF EUCLIDE ELEMENTS.

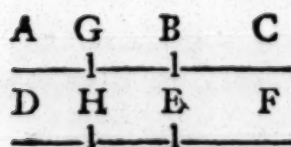
## PROPOSITION I. Plate VII. Fig. 15.

**A** Perpendicular-line DF drawn from D the center of a circle ABC to BC the side of a pentagon inscribed in the said circle, is the half of these two lines taken together, viz. of the side of the hexagon DE, and side of the decagon EC inscribed in the same circle ABC.

Take FG = FE, and draw CG; (a) Then CE is CG. Therefore the angle CGE (b) = CEG (b) = ECG therefore the angle ECG (c) = EDC (d) =  $\frac{1}{4}$  ADC =  $\frac{1}{2}$  CED ( $\frac{1}{2}$  ECD;) (f) consequently the angle GCE = ECG = EDC, (g) wherefore DG = GC (CE therefore DF = CE (DG) + EF =  $\frac{DE + CE}{2}$  which was to be demonstrated.

- a 4. 1.
- b 5. 1.
- c 32. 1.
- d hyp. and 33. 6.
- e 20. 3.
- f 7. ax.
- g 6. 1.

## PROP. II.



If two right-lines AB, DE, cut according to extreme and mean proportion (AB : AG :: AG : C and DE : DH :: DH : HE.) shall be cut after the same manner viz. in the same proportions (AG : GB :: DH : HE.)

Take BC = BG; and EF = EH. Then AB x BC (a) AGq. wherefore ACq (b) = 4 ABG + AGq (c) = AGq. In like manner shall DFq = 5 DHq. (d) therefore AC : AG :: DF : DH. whence by compounding AG : AG :: DF + DH : DH. that is, 2 AB : AG : DE : DH, (e) consequently AB : AG :: DE : DH; whence by division AG : GB :: DH : HE. Which was to be demonstrated.

- a 17. 6.
- b 8. 2.
- c 1. ax. 1.
- d 22. 5. and 22. 6.
- e 22. 5.
- f 17. 5.

P R C



PROP. III. Plate VII. Fig. 16, 17.

The same circle ABD comprehends both ABCDE the pentagon of a Dodecaedron, and LMN the triangle of an Icosaedron inscribed in the same sphere.

Draw the diameter AG, and the right-lines AC, CG; and let IK be the diameter of the sphere, (a) and IKq = OPq. (b) and make OP : OQ :: OQ : QP. Because ACq + CGq (c) = AGq (d) = 4 FGQ; and ABq (e) = CGq + CGq. (f) thence ACq + ABq = 5 FGq; moreover, because CA : AB (g) :: AB : CA - AB; and OP : OQ :: OQ : QP; (h) and as CA : OP :: AB : OQ (k) therefore 3 ACq (l) (IKq.) : 5 OPq (m) (IKq) :: 3 ABq : 5 OQq. therefore 3 ABq = 5 OQq. But because ML is the side of a pentagon inscribed in a circle, whose radius is OP, thence 15 RMq (n) = 5 MLq (p) = 5 OPq + 5 OQq = 3 ACq + 3 ABq (q) = 15 FGq; (r) therefore RM = FG. (s) and consequently the circle ABD is = the circle LMN. Which was to be demonstrated.

a sch. 47. 1.  
b 30. 6.  
c 47. 1.  
d 4. 2.  
e 10. 13.  
f 2. and  
3 ax.  
g 8. 13.  
h 2. 13. &  
16. 5.  
k 22. 6. &  
4. 5.  
l 15. 13.  
m constr.  
n cor. 16.  
13.  
o 12. 13.  
p 10. 13.  
q 15. 5.  
and above  
\* before  
r 1. ax. 1.  
and schol.  
48. 1.  
f 1. def. 3.

PROP. IV. Fig. 18, 19.

If from F the center of a circle encompassing the pentagon of a Dodecaedron ABCDE, a perpendicular-line FG be drawn to one side of the Pentagon CD; the rectangle contained under the said side CD and the perpendicular FG, being thirty times taken, is equal to the superficies of a Dodecaedron. Also,

If from the center L of a circle inclosing the triangle of an Icosaedron HIK, a perpendicular-line LM be drawn to one side of the triangle HK; the rectangle contained under the said side HK and the perpendicular LM, being thirty times taken, shall be equal to the superficies of an Icosaedron.

Draw FA, FB, FC, FD, FE; (a) then shall the triangles CFD, DFE, EFA, AFB, BFC be equal, but CD x FG (b) = 2 triangles CFD; therefore 30 CD x GF (c) = 60 CFD (d) = 12 pentagons ABCDE (e) = to the superficies of a Dodecaedron. Which was to be demonstrated.

Draw LI, LH, LK; then HK x LM (f) = 2 triangles LHK; therefore 30 HK x LM (g) = 60 HLK = 20 HIK (h) = the superficies of an Icosaedron. Which was to be demonstrated.

a 8. 1.  
b 41. 1.  
c 15. 5.  
d 6. ax.  
e 17. 3.  
f 41. 1.  
g 15. 5.  
h 6. 13.

## Coroll.

k 15. 5.

$CD \times FG : HK \times LM (k) ::$  the superficies of a Dodecaedron to the superficies of an Icosaedron.

## P R O P. V. Plate VII. Fig. 20.

*The superficies of a Dodecaedron bath to the superficies of an Icosaedron inscribed in the same sphere, the same proportion that H the side of a cube bath to AD the side of an Icosaedron.*

a 3. 14.

Let the circle ABCD (a) enclose both the Pentagon of a Dodecaedron, and the triangle of an Icosaedron; whose sides are BD, AD; upon which from the center E let fall the perpendiculars EF, EGC; and draw CD.

b 9. 13.

c 1. 14.

d cor. 12,

13.

e 15. 5.

f cor. 17.

13.

g 2. 14.

h 1. 6.

k 7. 5.

l cor. 4. 14.

Because  $EC \perp CD : EC (b) :: EC : CD$ . thence  $EG (c) (\frac{1}{2} EC \times \frac{1}{2} CD) : EF (d) (\frac{1}{2} EC) (e) :: EF : EG - EF (\frac{1}{2} CD.)$  but  $H : BD (f) :: BD : H - BD (g)$  therefore  $H : BD :: EG : EF$ . consequently  $H \times EF = BD \times EG$ . wherefore since  $H : AD (b) :: H \times EF : AD \times EF (k)$  thence shall be  $H : AD :: BD \times EG : AD \times EF (l) ::$  the superficies of a Dodecaedron to the superficies of an Icosaedron. Which was to be demonstrated.

## P R O P. VI. Fig. 21.

*If a right-line AB be cut in extreme and mean proportion, then as the right-line BF, containing in power that which is made of the whole line AB, and that which is made of the greater segment AC, is to the right-line E containing in power that which is made of the whole line AB, and that which is made of the lesser segment BC; so is the side of the cube BG to the side of an Icosaedron BK inscribed in the same sphere with the cube.*

a cor. 17,

13.

b 12. 13.

c 4. 13.

d 15. 5.

e 2. 14.

f 22. 6.

In the circle, whose semidiameter is AB, inscribe BFGHI the pentagon of a Dodecaedron; and BKL the triangle of an Icosaedron, (a) wherefore BG shall be the side of a cube inscribed in the same sphere; therefore  $BKq (b) = 3 ABq$ ; and  $Eq (c) = 3 ACq$ . therefore  $BKq : Eq (d) :: ABq : ACq (e) :: BGq : BFq$ ; wherefore by permutation  $BGq : BKq :: BFq : Eq (f)$  whence  $BG : BK :: BF : E$ . Which was to be demonstrated.

P R O P.

PROP. VII.

*A Dodecaedron is to an Icosaedron, as the side of a Cube is to the side of an Icosaedron, inscribed in one and the same sphere.*

Because (a) the same circle comprehends both the pentagon of a Dodecaedron, and the triangle of an Icosaedron, (b) the perpendiculars drawn from the center of the sphere to the planes of the pentagon and triangle, shall be equal one to another. Therefore if the Dodecaedron and Icosaedron be conceived and divided into pyramids, right-lines being drawn from the center of the sphere to all the angles, the altitudes of all the pyramids shall be equal one to the other. Wherefore since the pyramids are (c) of equal height with the bases, and the superficies of the Dodecaedron is equal to twelve pentagons, and the superficies of the Icosaedron to twenty triangles, the Dodecaedron shall be to the Icosaedron, as the superficies of the Dodecaedron is to the superficies of the Icosaedron; (d) that is, as the side of the cube is to the side of the Icosaedron.

a 3. 14.

b 47. 1.

c 5. and 6.  
12.

d 5. 14.

PROP. VIII. Plate VII. Fig. 22, 23.

*The same circle BCDE comprehends both the square of the cube BCDE; and the triangle of the octaedron FGH inscribed in one and the same sphere.*

Let A be the diameter of the sphere. Because Aq (a) = 3 BCq (b) = 6 BIq; and also Aq (c) = 2 GFq (d) = 6 KFq; thence shall BI = KF; (e) therefore the circle CBED = GFH. Which was to be demonstrated.

a 15. 13.

b 47. 1.

c 14. 13.

d 12. 13.

e 1. def. 3.

*The End of the Fourteenth Book.*



## The FIFTEENTH BOOK

O F

EUCLID'S  
ELEMENTS.

## PROPOSITION I. Plate VII. Fig. 24.

**I**N a cube given ABGHDCFE to describe a pyramid AGECE.

From the angle C draw the diameters CA, CG, CE; and connect them with the diameters AG, GE, EA. All which are (a) equal among themselves, as being the diameters of equal squares: therefore the triangles CAG, CGE, CEA, EAG are equilateral and equal; and consequently AGECE is a pyramid, which insits upon the angles of the cube, and therefore (b) is inscribed in it. Which was to be done.

## PROP. II. Fig. 25.

*In a pyramid given ABDC to describe an Octaedron EGKIFH.*

(a) Bisect the sides of the pyramid in the points E, I, F, K, G, H, which join with the 12 right-lines EF, FG, GE, &c. All these are (b) equal one to the other; consequently the 8 triangles EHI, IHK, &c. are equilateral and equal, and so make (c) an Octaedron described (d) in the given pyramid. Which was to be done.

PROP.

PROP. III. Plate VII. Fig. 26.

In a cube given CHGBDEFA to describe an Octaedron NPQSOR.

Connect \* the centers of the squares N, P, Q, S, O, R with the twelve right-lines NP, PQ, QS, &c. which are (a) equal among themselves; and so make 8 equilateral and equal triangles: wherefore (b) the Octaedron NPQSOR (b) is inscribed in the cube. Which was to be done.

\* 8, 4,

a 4. 1.

b 31. and

27. def. 11.

PROP. IV. Fig. 27.

In an Octaedron given ABCDEF, to inscribe a cube.

Let the sides of the pyramid EABCD, whose base is the square ABCD, be bisected by the right-lines, LM, MN, NO, OL, which are (a) equal and (b) parallel to the sides of the square ABCD; (c) then the quadrilateral LMNO is a square. In like manner, if the sides of the square LMNO be bisected in the points G, H, K, I, and GH, HK, KI, IG connected, GHKI shall be a square. And if in the other 3 pyramids of the Octaedron, the centers of the triangles be in the same sort conjoined with right-lines, then other squares will be described like and equal to the square GHKI; wherefore six such squares shall make a cube, which shall be described within an Octaedron, (d) since its eight angles touch the eight bases of the Octaedron in their centers. Which was to be done.

a 4. 1.

b 2. 6.

c 29. def. 1.

d 31. def.

11.

PROP. V. Fig. 28.

In an Icosaedron given to inscribe a Dodecaedron.

Let ABCDEF be a pyramid of the Icosaedron, whose base is the pentagon ABCDE; and the centers of the triangles G, H, I, K, L; which connect with the right-lines GH, HI, IK, KL, LG. Then GHIKL shall be a pentagon of the Dodecaedron to be inscribed.

For the right-lines, FM, FN, FO, FP, FQ, passing through the centers of the triangles, (a) bisect their bases; (b) therefore the right-lines MN, NO, OP, PQ, QM, (c) are equal one to the other; (d) whence also the angles

a cor. 3. 3.

b 4. 1.

c 4. 1.

d 8. 1.

†

MFN

e 4. 1.  
f 12. 13.

MFN, NFO, OFF, PFQ, QFM are equal; therefore the pentagon GHIKL is equiangular, (e) and consequently equilateral, since FG, FH, FI, FK, FL (f) are equal. And if in the other eleven pyramids of the Isocaedron, the centers of the triangles be in like sort joined with right-lines, then will the pentagons be equal and like to the pentagon GHIKL be described. Wherefore 12 of such pentagons shall constitute a Dodecaedron; which also shall be described in the Icosaedron, seeing the twenty angles of the Dodecaedron consist upon the centers of the twenty bases of the Icosaedron. Whereby it appears that we have described a Dodecaedron in an Icosaedron given. *Which was to be done.*

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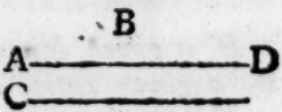
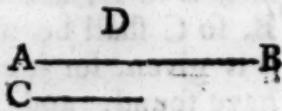
**F I N I S.**



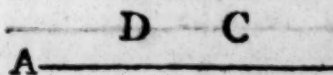
# EUCLIDE'S DATA.

## DEFINITIONS.

- I. **P**lanes or Spaces, Lines and Angels, to which we can find others equal, are said to be given in Magnitude.
- II. A Ratio is said to be given, when we can find it, or one equal to it.
- III. Rectiline-figures, whose angles are given, and also the ratio of the sides to one another, are said to be given in Species or Kind.
- IV. Points, Lines and Angles, which have and keep always one and the same place and situation, are said to be given in Position or Situation.
- V. A Circle is said to be given in Magnitude, when the semidiameter thereof is given in Magnitude.
- VI. A circle is said to be given in Position, and Magnitude, when the center thereof is given in Position, and the semidiameter in Magnitude.
- VII. Segments of Circles, whose angles and bases are given in Magnitude, are said to be given in Magnitude.
- VIII. Segments of a Circle, whose angles are given in Magnitude, and the bases of the segments in Position and Magnitude, are said to be given in Position and Magnitude.
- IX. A Magnitude AB, is greater than another Magnitude C, by a given Magnitude BD, when having taken away the given Magnitude DB, the rest AD, is equal to the other Magnitude C.
- X. A Magnitude AB, is less than another Magnitude C, by a given Magnitude BD, when having added thereto the given Magnitude BD, the whole AD is equal to the other Magnitude C.

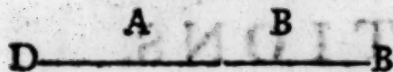


XI. A Magnitude AB is said to be greater than another Magnitude CB, by a given Magnitude AD, and in ratio, when



A ————— B taken from the same Magnitude the given Magnitude AD, the rest DB, hath to the other Magnitude CB, a given ratio.

XII. A Magnitude AB is said to be less than another Magnitude BC, by a given Magnitude AD, and in ratio, when the given magnitude AD being added thereto, the whole DB hath to the other Magnitude BC, a given ratio.



XIII. A right-line is said to be drawn down from a given point, unto a right-line given in Position, the right line being drawn in a given angle.

XIV. A right-line is said to be drawn up from a given point, to a right-line given in Position, the right-line being drawn in a given-angle.

XV. A right-line is against another right-line in Position, when it is drawn parallel thereto through a given point.

### PROPOSITION I.

TWO Magnitudes A and B being given, the ratio they have to one another A to B is also given.

*Demonstration.* For seeing that the Magnitude A is given, (a) we can find one equal thereto, which let be C. Again, forasmuch as the Magnitude B is given, we can also find one equal to that, and let that be D. Therefore seeing that A is equal to C, and B to D, as A is to C, (b) so is B to D, and by permutation, (c) as A shall be to B, so C shall be to D. Therefore (d) the ratio of A to B is given, for it is the same ratio as of C to D, as we have found, and which ought to be demonstrated.

### PROP. II.

If a given Magnitude A, hath to some other Magnitude B, a given ratio, that other Magnitude B, is also given in Magnitude.

*Demon.*

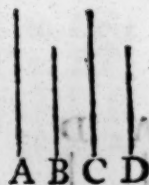
a 1. def.

b 7. 5.

c 16. 5.

d 2. def.

*Demonstr.* For seeing that A is given, we can find one equal thereto, which let be C: And forasmuch as the ratio of A to B, is also given, we can find (a) one of the same. Let it be found, and let the ratio be of C to D. Now seeing that as A is to B, so C is to D; and by permutation, as A is to C, so B is to D: But A is equal to C, therefore (b) B shall be also equal to D. Therefore (c) the Magnitude B is given, seeing that thereto there hath been found one equal, to wit, D.



a 2. def.

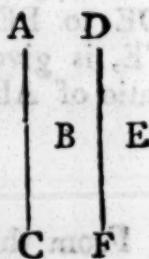
b 14. 5.

c 1. def.

P R O P. III.

*If given magnitudes AB and BC, are compounded, that magnitude AC, that is compounded of them, shall be also given.*

*Demonstr.* For seeing that AB is given, we can find one equal to it, which let be DE. Again, seeing that BC is given we can also find one equal to that, which let be EF. Wherefore seeing that DE is equal to AB, and EF is equal to BC, the whole AC (a) is equal to the whole DF. Therefore AC is given, seeing that DF is proposed equal thereto.

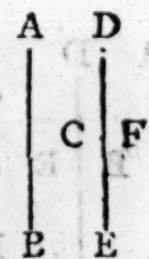


a 2. ax. 1.

P R O P. IV.

*If from a given magnitude AB, there be taken away a given magnitude AC, the remaining magnitude CB is also given.*

*Demonstr.* Forasmuch as AB is given, we can find one equal thereto, which let be DE. Again, seeing that AC is given, we can also find one equal to it, which let be DF. Seeing then that the Magnitude AB is equal to the Magnitude DE, and the Magnitude AC to the Magnitude DF; the rest CB (a) shall be equal to the rest FE. Wherefore CB is given, for to it there hath been found an equal, to wit, FE.

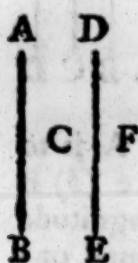


a 3. ax. 1.

P R O P.



## PROP. V.



If a Magnitude AB, hath a given ratio to some part thereof AC, it will have also a given ratio to the part remaining CB.

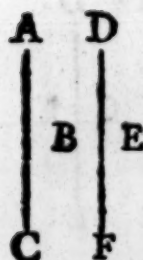
*Demonstr.* Let DE be exposed as a given Magnitude, and seeing that the ratio of the Magnitude AB, to Magnitude AC, is given (a) we can find one of the same, which let be DE to DF; therefore the ratio of the same

DE to DF is given. But DE being given, so is (b) also its part DF; and consequently, (c) the rest FE: Therefore (d) seeing that DE and FE are given, the ratio of the same DE to FE is also given. And forasmuch as DE is (e) to DF, as AB is to AC, and by conversion, as DE to FE, so AB is to CB. But the ratio of DE to FE, is given, as hath been demonstrated; therefore the ratio of AB to CB is also given.

*Scholium.*

From this it is evident that if a Magnitude hath to some part thereof a given ratio, by division, the ratio that one part hath to the other, shall be also given. For seeing that as DE is to FE, so is AB to CB; by division, as DF to FE, so AC to CB. But it hath been demonstrated that the parts DF and EF are given, and consequently their ratio is also given: In like manner, therefore, the ratio of AC to CB is given.

## PROP. VI.



If two Magnitudes AB and BC, having to one another a given ratio, are compounded, the Magnitude AC compounded of them, shall also have a given ratio to each of them AB and BC.

*Demonstr.* Let the given Magnitude DE be proposed, and seeing that the reason of AB to BC is given; let there be made one and the same of the said DE to EF; therefore the ratio

of the same DE to EF is given; and therefore (a) the Magnitude DE being given, both the one and the other of them DE and FE, is given. Wherefore (b) the whole DF shall be also given. Therefore (c) the ratio of the same DF to each of them DE and EF, shall be given.

And

a 2. def.

b 2. prop.

c 4. prop.

d 1. prop.

e 19. 5.

a 2. prop.

b 3. prop.

c 1. prop.

And forasmuch-then as AB is to BC, so is DE to EF; by compounding, (d) as AC is to BC, so is DF to EF: Therefore by conversion, as AC to AB, so is DF to DE. Therefore as the whole DF is to each of the other Magnitudes DE and FE, so the whole AC is to each of the Magnitudes AB and BC. Therefore (e) the ratio of the same AC to each of the Magnitudes AB and BC is given.

d 18. 5.

e 2 def.

PROP. VII.

If a given Magnitude AB be divided according to a given ratio AC to CB, A———C———B each segment AC and CB is given.

Demonstr. For seeing the ratio of AC to CB is given, the ratio of (a) AB to each of them (AC and CB) is also given. But AB is given: Therefore (b) each of the segments AC and CB is also given.

a 6. prop.  
b 2. prop.

PROP. VIII.

Magnitudes A and C, which have to one and the same a given ratio, B, shall be to one another in a given ratio A to C.

|   |   |
|---|---|
| A | D |
| — | — |
| B | F |
| — | — |
| C | E |
| — | — |

Demonstr. For let the given magnitude D be proposed, and seeing that the ratio of A to B is given, let the same be done of the said D to E. Now seeing that D is given, (a) E is also given. Again, seeing that the ratio of B to C is given, let the same be done of E to F. But E is given, and therefore F is also given. But seeing that D is given, (b) the ratio of the same D to F is given; and seeing that as A to B, so is D to E, and as B to C, so is E to F; in ratio of equality, (c) as A is to C, so is D to F; but the ratio of D to F is given. Therefore the ratio of A to C is also given.

a 2. prop.

b 1. prop.

c 22. 5.

PROP.

## PROP. IX.

|          |          |
|----------|----------|
| <u>A</u> | <u>D</u> |
| <u>B</u> | <u>E</u> |
| <u>C</u> | <u>F</u> |

If two or more magnitudes A, B, and C, are to one another in a given ratio, and that the same magnitudes A, B, and C, have to other magnitudes D, E, and F, given ratio's, although they be not the same, those other magnitudes D, E, and F shall be also to one another in given ratio's.

*Demonstr.* Forasmuch as the ratio of A to B is given, as also that of A to D, the ratio of D to B shall be given: But the ratio of B to E is also given; therefore the ratio of the same D to E shall be in like manner given. Again, seeing that the ratio of B to C is given, and also that of B to E, the ratio of E to C shall be given. But the ratio of C to F is also given. Therefore (a) the ratio of E to F shall be given. But it hath been demonstrated that the ratio of D to E is also given; and therefore (b) the ratio of D to F shall be given. Therefore the magnitudes D, E, and F are to one another in given ratio's.

a 8. prop.

b 8. prop.

## PROP. X.

A ——— D B C

If a magnitude AB, be greater than another magnitude BC, by a given magnitude, and in ratio, the magnitude AC compounded of both, shall be also greater than that the same magnitude, by a given magnitude, and in ratio; but if that compounded magnitude be greater than the same magnitude, by a given magnitude, and in ratio; either the remainder shall be also greater than that same, by a given magnitude, and in ratio; or else the same remainder is given with the following, to which the other magnitudes hath a given ratio.

*Demonstr.* For seeing that AB is greater than BC by a given magnitude, and in ratio, let the given magnitude AD be taken away. Therefore (a) the reason of the remainder DB to BC is given; and by compounding, (b) the ratio of DC to BC is also given. But the magnitude AD is also given; therefore AC is greater than the same BC by a given magnitude, and in ratio.

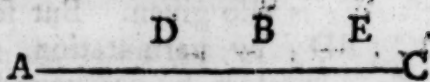
Again,

a 11. def.

b 6. prop.



Again, Let the magnitude AC be greater than the magnitude BC, by a given magnitude, and in ratio; I say, that the rest AB, is either greater than the same BC by a given magnitude, and in ratio; or that the same AB, with that which followeth, to which BC hath a given ratio, is given.



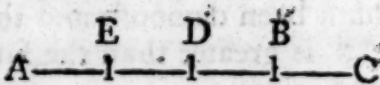
Forasmuch as the magnitude AC is greater than the magnitude BC, by a given magnitude, and in ratio, cut off from it the given magnitude: Now the same given magnitude is either less than the magnitude AB, or greater: Let it in the first place be less, and let it be AD. Therefore the ratio of the remainder DC to CB is given. Wherefore by division, the ratio of DB to BC is given. But the magnitude AD is also given; therefore the magnitude AB is greater (c) than the magnitude BC by a given magnitude, and in ratio. Now let the given magnitude be greater than the magnitude AB, and let AE be put equal thereto; therefore (d) the ratio of the remainder EC to CB is given; and by conversion, (e) the ratio of the same BC to BE, is also given. But the same EB with BA is given, for that the whole AE is given: Therefore there is given AB, with that which follows BE, to which BC hath a given ratio.

c 11. def.

d 11. def.  
e 5. prop.

PROP. XI.

If a magnitude AB be greater than a magnitude BC, by a given magnitude, and in ratio, the same magnitude AB, shall be also greater than the magnitude compounded of them by a given magnitude, and in ratio, and if the same magnitude be greater than the two others together by a given magnitude; and in ratio, that same magnitude shall be also greater than the rest by a given magnitude, and in ratio.



Demonstr. For seeing that the magnitude AB is greater than BC by a given magnitude and in ratio; let there be taken from it a given magnitude AD: Therefore (a) the ratio of the rest DB to DC, is given; and therefore (b) the ratio of DC to BD shall be also given: Let the same be done of AD to DE, therefore the ratio of the same AD to DE is given. But AD is given; therefore (c) DE is also given, and consequently; (d) the rest

a 11. def.  
b 6. prop.

c 2. prop.  
d 4. prop.

R.

rest

e 16. 5.  
f 18. 5.  
g 16. 5.

h 11. def.

i 11. def.

k 5. prop.

l 2. prop.

m 19. 5.

n schol. 5.  
prop.

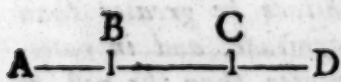
o 11. def.

rest AE, is also given. But seeing that as  $AD : DE :: DC : BD$ ; by permutation, (e) as  $AD : DC :: DE : DB$ ; Therefore by compounding, (f) as  $AC : CD :: EB : DB$ ; and by permutation, (g) as  $AG : EB :: DC : DB$ . But the ratio of DC to DB is given: Therefore also is AC to EB, and consequently that of EB to AC. But it hath been demonstrated that AE is given, therefore (b) AB is greater than AC by a given Magnitude, and in ratio.

But now let AB be greater than AC by a given Magnitude, and in ratio: I say, that the same AB is also greater than the rest BC by a given Magnitude, and in ratio.

For seeing that AB is greater than AC by a given Magnitude, and in ratio. Let the given Magnitude AB be cut off therefrom: Therefore (i) the ratio of the remainder EB to AC is given, and consequently also shall be given that of AC to EB. Let the same be done of AD to DE, therefore the ratio of AD to DE is given; and by conversion, (k) the ratio of AD to AE shall be also given, and consequently that of AE to AD. Now AE is given, therefore the whole AD (l) shall be also given; and seeing that as the whole AC is to the whole EB, so the part cut off AD, is to the part cut off ED, so also shall be (m) the remainder DC to the remainder DB. But the ratio of AC to EB is given: Therefore also shall be given that of DC to DB. Wherefore by division (n) the ratio of BC to DB is given; and consequently also shall be given that of DB to BC. But it hath been demonstrated that AD is given: Therefore (o) AB is greater than the same BC by a given Magnitude, and in ratio.

### P R O P. XII.



If there are three magnitudes AB, BC, and CD; and that the first AB, with the second BC, to wit, AC, be given. And the second BC, with third CD, to wit, BD, be also given: Either the first AB shall be equal to the third CD, or the one shall be greater than the other by a given Magnitude.

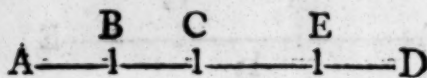
Demonstr. Forasmuch as each of the Magnitudes AC and BD are given, the given Magnitudes are either equal to one another, or unequal. Let them be first equal. Therefore AC is equal to BD; take away the common Magnitude

Magnitude BC, and there will remain (a) AB, equal to CD. But suppose them to be unequal as in this second figure, and let BD be greater than AC: Let then BE be put equal to AC.

Now seeing that AC is given, BE is also given. But the whole BD is also given, the rest ED (b) shall be so also; and forasmuch as BE is equal to AC, taking away the common Magnitude (c) BC, there will remain AB equal to CE. But ED is given; Therefore CD is greater than AB by the given Magnitude ED.

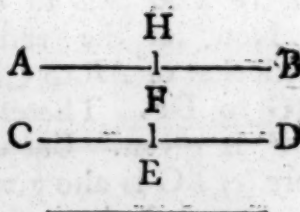
a 3. ax. 1.

b 4. prop.  
c 3. ax. 1.



P R O P . XIII.

If there are three magnitudes AB, CD, and E, and that the first of them AB, hath a given ratio to the second CD; but the second CD is greater than the third E, by a given magnitude, and in ratio; also the first AB, shall be greater than the third E, by a given magnitude, and in ratio.



*Demonstr.* For seeing that CD is greater than E by a given Magnitude, and in ratio; let the given Magnitude CF be taken therefrom: Therefore the ratio of the rest FD to E is given. And forasmuch as the ratio of AB to CD is given, let the same be done of AH to CF. Therefore the ratio of the same AH to CF is given. But CF is given: Therefore (a) AH is also given. And seeing that as the whole AB is to the whole CD, so is the part cut off AH is to the part cut off CF, and so (b) also the rest HB is to the rest FD, the ratio of the same HB to FD is also given. But the ratio of FD to E is also given: Therefore (c) the ratio of HB to F is given. But it hath been demonstrated that AH is given: Therefore (d) AB is greater than the said E by a given Magnitude, and in ratio.

a 2. prop.

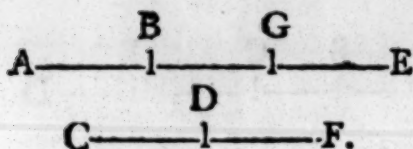
b 19. 5.

c 8. prop.

d 11. def.



## PROP. XIV.



If two Magnitudes AB and CD, have to one another a given ratio, and that to each of them there be added a given Magnitude,

to wit, BE and DF; either the whole AE and CF shall have to one another a given ratio, or the one shall be greater than the other by a given Magnitude, and in ratio.

a 1. prop.

b 12. 5.

*Demonstr.* For seeing that each of those Magnitudes BE and DF, is given, (a) the ratio of the said BE and DF is also given; and if that ratio be the same with that of AB to CD, that of the whole AE to the whole CF, (b) shall be the same; and therefore the ratio of the said AE to CF is given.

c 2. prop.

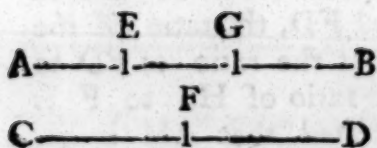
d 4. prop.

e 12. 5.

f 11. def.

Now let the ratio of BE to DF be not the same with that of AB to CD, and let it be as AB to CD, so is BG to DF. Therefore the ratio of the said BG to DF is given. But the Magnitude DF is given, therefore (c) BG is also given; and seeing that the whole BE is given, (d) the rest GE shall be also given. But forasmuch as AB is to CD, as BG is to DF, (e) so also is the whole AG to the whole CF; and therefore the ratio of the said AG to CF is given: But the Magnitude GE is given: Therefore (f) the Magnitude AE is greater than the Magnitude CF by a given Magnitude, and in ratio.

## PROP. XV.



If two Magnitudes AB and CD, have to one another a given ratio, and that from each of them be taken away a given Magnitude (to wit, from the

Magnitude AB the Magnitude AE, and from the Magnitude CD the Magnitude CF) the remaining Magnitudes EB and FD, either shall have to one another, a given ratio, or the one of them shall be greater than the other by a given Magnitude, and in ratio.

†

*Demonstr.*

*Demonstr.* For seeing that each Magnitude AE and CF is given, the ratio of AE to CF is given; and if it be the same with that of AB to CD, that of the remainder EB to the remainder FD, (a) shall be also the same; and therefore the ratio of the said EB to FD shall be also given. But if it be not the same, let it be as AB to CD, so is AC to CF. Now the ratio of AB to CD is given, therefore also that of AG to CF shall be given. But CF is given, therefore (b) AG is given. But AE is also given, therefore (c) the rest EG is given; and seeing that as AB is to CD, so the part cut off AG is to the part cut off CF, and so also is (d) the rest GB to the rest FD; the ratio of the said GB to FD is also given. Therefore seeing that EG is given, EB is greater than FD (e) by a given Magnitude, and in ratio.

a 19. 5.

b 2. prop.

c 4. prop.

d 4. prop.

e 11. def.

PROP. XVI.

If two Magnitudes AB and CD, have to one another a given ratio, and that from one of them, to wit, CD, there be taken away a given Magnitude DE, and to the other AB there be added a given Magnitude BF, the whole AF shall be greater than the rest CE, by a given Magnitude, and in ratio.

*Demonstr.* For seeing that the ratio of AB to CD is given, let the same be made of BG to DE: Therefore (a) the ratio of the said BG to DE is given. But DE is given, therefore (b) BG is also given. But BF is also given, therefore (c) the whole GF is given. And seeing that as AB is to CD, so the part cut off BG, is to the part cut off DE; and (d) so also is the remainder AG to the remainder CE; the ratio of the said AG to CE is given, But GF is given, therefore the Magnitude AF is greater than the Magnitude CE by a given Magnitude, and in ratio.

a 2. def.

b 2. prop.

c 3. prop.

d 19. 5.

## R P O P. XVII.



*If there are three Magnitudes AB, E, and CD, and that the first AB be greater than the second E by a given Magnitude, and in ratio: And the third CD be also greater than the same second E, by a given Magnitude, and in ratio; the first AB shall have to the third CD, either a given ratio, or else the one shall be greater than the other by a given Magnitude, and in ratio.*

*Demonstr.* For seeing that AB is greater than E by a given Magnitude, and in ratio, let the Magnitude AF be taken away: Therefore the ratio of the remainder FB to E is given. Again, seeing that CD is greater than the said E by a given Magnitude, and in ratio, let the given Magnitude CG be cut off therefrom; and the ratio of the remainder GD to E shall be given: Therefore (a) the ratio of FB to GD shall be also given. But to the said FB and GD are added the given Magnitudes AF and CG: Therefore the whole AB and CD (b) shall either have to one another a given ratio, or the one shall be greater than the other by a given Magnitude, and in ratio.

a 3. prop.

b 14. prop.

## P R O P. XVIII.



*If there are three Magnitudes AB, CD, and EF, and that the one of them, to wit, CD, be greater than either of the other AB or EF, by a given Magnitude, and in ratio; either the two others AB and EF, shall have to one another a given ratio, or the one shall be greater than the other by a given Magnitude, and in ratio.*

*Demonstr.* Forasmuch as the Magnitude CD is greater than the Magnitude AB by a given Magnitude, and in ratio, let the given Magnitude LG be taken therefrom: Therefore the ratio of the remainder CG to AB is given. Let the same be made of GD to BH, therefore the ratio of the said DG to BH is given. But DG is given, therefore (a) BH is also given. And seeing that as CG is to AB, so is GD to BH, (b) so also is the whole CD to the whole AH, the ratio of the said CD to AH shall be also given.

a 2. prop.

b 12. 5.

Again,



Again, seeing that the same CD is greater than EF by a given Magnitude, and in ratio; let the Magnitude DI be cut off therefrom: Therefore the ratio of the remainder CI to EF is given: Let the same be made of DI to FK. Therefore the ratio of the said DI to FK shall be also given. But DI is given, therefore FK is also given. And seeing that as CI is to EF, so is ID to FK; so also is the whole (c) CD to the whole EK; the ratio of the said CD to FK shall be given. But the ratio of the same CD to AH is also given: Therefore (d) the ratio of the said AH to EK shall be given. And seeing that from the said AH and EK, the given Magnitudes BH and FK are cut off, the Magnitudes AB and EF (e) are either in a given ratio to one another, or the one is greater than the other by a given Magnitude, and in ratio.

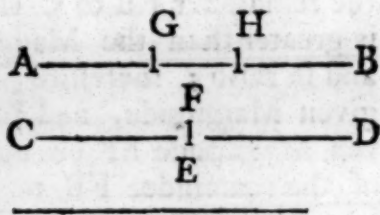
c 12. 5.

d 8. prop.

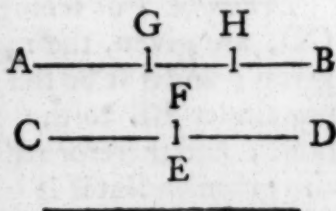
e 15. prop.

PRO P. XIX.

If there are three Magnitudes AB, CD, and E, and that the first AB be greater than the second CD, by a given Magnitude, and in ratio; and that the second CD be greater than the third E, by a given Magnitude, and in ratio; also the first Magnitude AB shall be greater than the third E, by a given Magnitude, and in ratio



*Demonstr.* For seeing that CD is greater than E by a given Magnitude, and in ratio; let the given Magnitude, CF be taken therefrom: Therefore the ratio of the remainder FD to E is given. Again, seeing that AB is greater than the same CD by a given Magnitude, and in ratio: Let the Magnitude AG be taken therefrom: Therefore the ratio of the remainder GB to CD is given: Let the same be made of GH to CF. Therefore the ratio of the said GH to CF is given. But CF is given: Therefore also GH is given, and then AG is also given, the whole (a) AH shall be also given. But as GB is to CD, so is GH to CF, and so also (b) the remainder HB to the remainder FD: Therefore the ratio of the said HB to FD is given. But the ratio of the same FD to E is also given: Therefore the ratio of HB to E is in



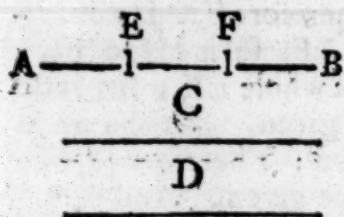
a 3. prop.

b 13. 3.

e 11. def.

given, and so is also the Magnitude AE: Wherefore the Magnitude AB (c) is greater than E by a given Magnitude, and in ratio.

## OTHERWISE.



*Construction.* Let there be three Magnitudes AB, C, and D, and let AB be greater than C by a given Magnitude, and in ratio; but let C be also greater than D, by a given Magnitude, and in ratio:

I say, that AB is greater than D by a given Magnitude, and in ratio.

*Demonstr.* Forasmuch as AB is greater than C by a given Magnitude, and in ratio, let the given Magnitude AE be cut off therefrom: Therefore the ratio of the remainder EB to C is given. But the Magnitude C is greater than the Magnitude D by a given Magnitude, and in ratio; therefore (d) EB is greater than D by a given Magnitude, and in ratio: Wherefore let the given Magnitude EF be cut off therefrom; and the ratio of the remainder FB to D shall be given. But AF is (e) given. Therefore AB is greater than D by a given Magnitude, and in ratio.

d 13. prop.

e 3. prop.

## PROP. XX.



*If there are two given Magnitudes, AB and CD, and that from them there are taken Magnitudes AE and CF, having to one another a given ratio; either the remaining Magnitudes EB and FD, shall have to one another given ratio's; or else the one shall be greater than the other by a given Magnitude, and in ratio.*

*Demonstr.* For seeing that both the Magnitudes AB and CD, are given, the ratio of the said AB to CD is (a) also given; and if it be the same as of AE to CF, that of the remainder EB to the remainder FD shall be (b) also the same; and therefore the ratio of the said EB to FD shall be also given. But if it be not the same, let it be so as that AE be to CF, as AG to CD. Now the ratio of the said AE to CF is given: Therefore the ratio of the said AG

a 1. prop.

b 19. 5.

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AG to CD is given. But CD is given, therefore (c) AG is also given. But the whole AB is likewise given, therefore (d) the remainder BG is given. And seeing that as AE is to CF, so is AG to CD, and also the remainder EG to the remainder FD, the ratio of the said EG to FD is given. But GB is also given: Therefore the Magnitude EB is greater (e) than the Magnitude BD by a given Magnitude, and in ratio.

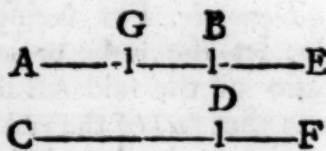
c 3. prop.

d 4. prop.

e 11. def.

## PROP. XXI.

If there are two Magnitudes given, AB and CD, and to them are added other Magnitudes BE and DF, having to one another a given ratio; either the whole AE and CF shall have to one another a given ratio, or else the one shall be greater than the other by a given Magnitude, and in ratio.



*Demonstr.* For seeing that both the Magnitudes AB and CD are given: their ratio (a) is also given, and if it be the same ratio as of BE to DF, the ratio of the whole AE to the whole CF shall be also given; for it shall be (b) the same. But if it be not the same, let it be as BE is to DF, so BG to CD: Therefore the ratio of the said BG to CD is given. But CD is given; therefore (c) also BG shall be given. But the whole AB is given; therefore also the (d) remainder AG shall be given. And seeing that as BE is to DF, so is BG to CD, and also (e) the whole GE to the whole CF, the ratio of the said GE to CF shall be likewise given. But AG is given; therefore the Magnitude AE is greater than the Magnitude CF by a given Magnitude, and in ratio.

a 1. prop.

b 12. 5.

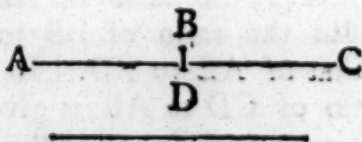
c 2. prop.

d 4. prop.

e 12. 5.

## PROP. XXII.

If two Magnitudes AB and BC, have to some other Magnitude D, a given ratio, also their compound Magnitude AC, shall have to the same Magnitude D, a given ratio.



*Demonstr.* For seeing that each Magnitude AB and BC hath a given ratio to D, the ratio (a) of AB to BC is given; and by compounding, (b) the ratio of AC to BC is given. But that of BC to D is also given, therefore (c) the ratio of the said AC to D shall be likewise given.

a 8. prop.

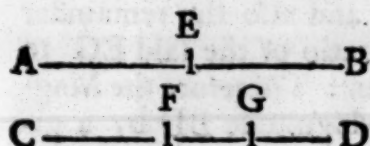
b 6. prop.

c 8. prop.

PROP.

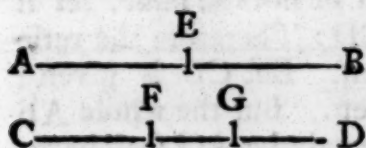


## PROP. XXIII.



If the whole AB be to the whole CD in a given ratio, and that the parts AE and EB be to the parts CF and FD in given ratio's, altho' they be not the same, the whole (to wit, AB, AE, and BE,) shall be to the whole (to wit, CD, CF, and FD,) in given ratio's.

*Demonstr.* For seeing that AE is to CF in a given ratio, let the same be made of AB to CG; therefore the ratio of the said AB to CG is given; and consequently also that (a) of the rest EB to the rest FG. But the ratio of FD to the same EB is also given: Therefore the ratio of FD to FG (b) is likewise given, and therefore (c) that of FD to the remainder GD is also given. But the ratio of AB to each of the Magnitudes CD and CG is given: Therefore (d) also the ratio of CD to CG is given, and again (e) that of CD to the remainder GD. But the ratio of FD to DG is given, therefore also (f) that of the same CD to FD, and consequently that of (g) CD to the remainder FC; and therefore



also the ratio of CF to FD shall be given. But the ratio of EB to FD is proposed to be given; therefore the ratio of CF to EB shall be given. Again, for that

the ratio of AB to CD is given; and also that of CD to each of those FC and FD, the ratio of the same AB to each of the said FC and (b) FD, shall be likewise given. But the ratio of the said FD to EB is given: Therefore the ratio of AB to BE shall be also given, and consequently AB to the remainder (i) AE. Wherefore by division (k) the ratio of AE to EB shall be likewise given. But the ratio of EB to FD is given. Therefore also that of AE to FD. In like manner, seeing that the ratio of CD to AB is given; and that of AB to each of his parts AE and EB; also the ratio of the said CD to each of the said AE and EB, (l) shall be given: Wherefore each of the Magnitudes AB, CD, AE, EB, CF, and FD, is to each of the others in a given ratio.

a 19. 5.

b 8. prop.

c 5. prop.

d 8. prop.

e 5. prop.

f 8. prop.

g 5. prop.

h 8. prop.

i 5. prop.

k sch. 5. pr.

l 8. prop.

P R O P.

P R O P. XXIV.

If of three right-lines A, B, and C, proportional A to B, as B to C, the first A hath to the third C a given ratio, it will also have to the second B a given ratio.

|   |   |
|---|---|
| A | D |
| B | E |
| C | F |

*Demonstr.* For, let there be exposed another right-line D, and seeing that the ratio of A to C is given; let the same be made of D to F; therefore the ratio of D to F is given. But D is given, therefore F is also given; betwixt the two right-lines D and F, let there be taken (a) a mean proportional E. Therefore the rectangle made under D and F is equal (b) to the square of E. But the same rectangle of D and F is (c) given: (for all the angles of that rectangle are given, being right-angles, and the ratio's that the sides have to one another are also given;) therefore the square of E is given, and consequently the same right-line E is also given (for one equal thereto may be found, (d) seeing that the rectangle of D and F is given.) But D is given, therefore (e) the ratio of D to E is given, and as A is to C, so D is to F. But as A is to C, (f) so is the square of A to the rectangle of A and C, and also as D is to F, so is the square of D to the rectangle of D and F. Therefore as the square of A is to the rectangle of A and C, so is the square of D to the rectangle of D and F. But the rectangle of A and C is equal to the square of B, (seeing that A, B, and C, are proportional) and that of D and F to the square of E, therefore as the square of A is to the square of B, so is the square of D to the square of E: Wherefore (g) as A is to B, so is D to E. But the ratio of D to E is given, therefore (b) also the ratio of A to B is given.

a 13. 7.

b 17. 6.

c 3. def.

d 14. 2.

e 1. prop.

f 1. 6.

g 12. 6.

h 2. def.

P R O P. XXV. Plate VII. Fig. 29.

If two lines AB and CD, given by position do intersect, the point E in which they cut one another, is given by position.

*Demonstr.* For if it change its place the one or the other of the lines AB and CD, would change its position: But so it is that by Supposition it changeth not: Therefore (a) the point E is given by position.

a 4. def.

P R O P.

## P R O P. XXVI.

**A——B** *If the extremities A and B, of a right-line AB, are given in position, that same right-line AB is given in position and in magnitude.*

*Demonstr.* For if the point A remaining in its place, the position, or the magnitude of the right-line AB shall change, the point B will fall elsewhere. But so it is, that by Supposition it doth not fall elsewhere. Therefore the right-line AB is given in position, and in magnitude.

## P R O P. XXVII.

**A——B** *If one of the extremes A of a right-line AB, given in position and magnitude, be given, the other extremity B shall be also given.*

*Demonstr.* For if the point A remaining in its place, the point B shall change and fall in some other place, either the position of the right-line AB, or its magnitude would change: But so it is that according to the Supposition, neither the one nor the other doth change. Therefore the point B is given.

## O T H E R W I S E.

*Constr.* On the center A, (Fig. 30.) with the distance AB, describe the circumference BC.

a 6. def.

b 25. prop.

*Demonstr.* Therefore (a) that circumference BC is given by position. But the right-line AB is also given by position; therefore the point (b) B is given.

## P R O P. XXVIII. Plate VII. Fig. 31.

*If through the given point A, there be drawn a right-line DAE, against another right-line BC, given in position, the right-line DAE so drawn, is given in position.*

*Demonstr.* For if it be not given, the point A remaining in its place, the position of the right-line DAE may change: Let it then change if it be possible, and fall elsewhere, remaining parallel to BC, and let it be the line FAG: Therefore BC is parallel to the said line FAG,

FAG.  
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which  
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and le  
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FAG. But (a) the same BC is also parallel to DAE : Therefore (b) DAE is parallel to the said line FAG, which is absurd ; seeing they join together, and meet in A : Therefore the position of the right-line DAE falls not elsewhere. Wherefore the said line DAE is given in position.

a 13. def.

b 30. 1.

P R O P. XXIX. Plate VII. Fig. 32.

If to a right-line AB, given in position, and to a point C given therein, there be drawn a right-line CD, which shall make a given angle ACD, the line drawn CD is given in position.

Demonstr. For if it be not given in position, the point C remaining in its place, the position of the line CD observing the Magnitude of the angle ACD, will fall elsewhere. Let it fall elsewhere then if it be possible, and let it be CE. Therefore the angle ACD is equal to the angle ACE, the greater to the lesser, which is absurd. Therefore the position of the right-line CD, shall not fall elsewhere ; and therefore the said line CD is given in position.

P R O P. XXX. Fig. 33.

If from a given point A, be drawn to a right-line BC, given in position, a right-line AD, making a given angle ADB, that line drawn AD is given in position.

Demonstr. For if it be not given, the point A remaining in its place, the position of the right-line AD changing, the magnitude of the angle ADB, will change. Let it change then, and let it be the right-line AE : Therefore the angle ADB is equal to the angle AEB, the greater (a) to the lesser, which is absurd. Therefore the position of the right-line AD doth not change ; and therefore the said line AD is given in position.

a 16. 1.

O T H E R W I S E.

Constr. Through the point A let there be drawn the line EAF, parallel to the right-line BC.

Demonstr. Then seeing that through the given point A, and against the right-line BC, given in position, there is drawn

b 29. 1.

drawn the right-line EF, (Fig. 34.) those lines EF and BC are parallels. But on the same lines doth also fall the right-line AD. Therefore (b) the angle FAD is equal to the given angle ADB; and therefore it is also given. Wherefore to the right-line EF given in position, and to the given point A therein, there is drawn the right-line AD, making the given angle FAD. Therefore (c) the said line AD is given in position.

c 29. prop.

## O T H E R W I S E.

*Constr.* In the line BCE, (Fig. 35.) let there be taken the given point C, and through the same let there be drawn the line CF, parallel to the said DA.

d 29. 1.

*Demonstr.* Forasmuch as AD and CF are parallels, and that on them there falls the right-line BCE, the angle FCB is equal (d) to the given angle ADB; and therefore it is also given. And seeing that the right-line BC is given in position, and that to a given point C therein, there is drawn the right-line FC, making the given angle FCB, that same line FC (e) is given in position. But through the given point A, opposite to the line FC given in position, there is drawn the line AD. Therefore the said line (f) AD is given in position.

e 29. prop.

f 28. prop.

## O T H E R W I S E.

*Constr.* In the right-line BC (Fig. 36.) assume some point at F, and draw AF.

g 26. prop.

h 32. 1.

i 29. prop.

*Demonstr.* Forasmuch as each point A and F is given, the right-line AF is given (g) in position. But the line BC is also given in position. Therefore \* the angle AFD is given. But by supposition, the angle ADF is given: Therefore DAF (which is the residue (h) of two right-angles) is given; and seeing that to the right-line AF given in position, and to the given point therein A there is drawn the right-line DA, making the given angle DAF, (i) that same line DA is given in position.

## Scholium.

\* EUCLIDE supposeth here that two right-lines being given in position, and inclining to one another, make a given angle, which some demonstrate after this manner.

Demon. &amp;c.

*Demonstr.* Forasmuch as the two right-lines given in position, incline to one another, the inclination of those lines is given. But the angle is the inclination of the lines: Therefore the angle which makes the right-lines given in position, and inclining to one another, is given.

*Another thus demonstrates it.*

*Constr.* Let there be two right-lines inclining to one another, as AB and CB, (Fig. 37, 38.) given in position, and in the line AB let there be taken a given point A, and in BC also some point, as C; and let the right-line AC be drawn.

*Demonstr.* Seeing that as well the point B, as each of the points A and C is given, (k) the three right-lines AB, BC, and AC, are given in Magnitude. Wherefore of three direct lines equal unto them, a triangle may be constituted: Let there then be made the triangle FDE, having the side FD equal to the side AB, the side FE equal to the side AC, and the base DE equal to the base BC.

k 26. prop.

Seeing then the angles comprised of equal right-lines are equal, we have found the angle FDE equal to the angle ABC; and therefore the same (l) angle ABC is given.

l 1. def.

PROP. XXXI. Plate VII. Fig. 39.

*If from a given point A there be drawn to a right-line given in position BC, a right-line AD, given in magnitude, that line AD shall be also given in position.*

*Constr.* From the center A, with the distance AD, let the circle DEF be described.

*Demonstr.* Forasmuch as the center A is given in position, and the semidiameter AD in magnitude, the circle DEF (a) is given in position. But the right-line BC is also given in position: Therefore the point of intersection D (b) is given, and seeing that the point A is also given; (c) the right-line AD is given in position.

a 6. def.

b 25. prop.

c 26. prop.

PROP. XXXII. Fig. 40.

*If unto parallel right-lines AB and CD, given in position, there be drawn a right-line EF, making the given angles BEF and EFD, the line drawn EF shall be given in magnitude.*

+

*Constr.*



*Constr.* For let there be taken in the line CD a given point G, and from that point let be drawn GH parallel to FE.

a 29. 1.

b 29. prop.

c 25. prop.

d 26. prop.

e 34. 1.

f 1. def.

*Demonstr.* Forasmuch as the lines EF and HG are parallels, and that on them falls the line CD; (a) the angle EFD is equal to the angle FGH. But the angle EFD is given, therefore the angle FGH is also given. And forasmuch as to the right-line CD given in position, and to the point G given in the same, there is drawn the right-line GH, making the given angle FGH, (b) the said line GH is given in position. But AB is also given in position, therefore (c) the point H is given. But the point G is also given: Therefore (d) the line GH is given in Magnitude, and is (e) equal to EF. Wherefore (f) the said line is given in Magnitude.

P R O P. XXXIII. Plate VII. Fig 41.

*If unto parallel right-lines AB and CD, given in position, there be drawn a right-line EF given in magnitude, that line EF shall make the given angles BEF and DFE.*

*Constr.* For let there be taken in the right-line AB the point G, and through that point let there be drawn the line GH parallel to EF.

a 34. 1.

b 6. def.

c 25. prop.

d 26. prop.

e sch. 30.

prop.

f 29. 1.

g 29. 1.

*Demonstr.* Therefore EF is equal to the said (a) GH. But EF is given in Magnitude, therefore GH is also given in Magnitude. But the point G is given, and therefore if on that point, with the distance GH, there be described a circle, (b) that circle shall be given in position: Let it be then described, and let it be HKL, the said circle HKL is therefore given in position. But the line CD which cuts the circumference KHI, in H, is also given in position. Therefore the said point of intersection H (c) is given. But the point G is given: Therefore (d) the right-line GH is given in position. But the right-line CD is also given in position: Therefore (e) the angle GHF is given. But to that angle (f) the angle EFD is equal; Therefore the angle EFD is given; and therefore also the angle BEF; for that is only the residue of the sum of two (g) right-angles.

O T H E R W I S E.

*Constr.* Let there be taken in the right-line CD; (Fig. 42.) the point G, and let GD be put equal to EF, then from the center G, with the distance GD, let there be described the circle HDB, and draw GB.

*Demonstr.*

*Demonstr.* Forasmuch as the center  $G$  is given in position, and the semidiameter  $GD$  in magnitude, the circle  $BDH$  ( $b$ ) is given in position. But the line  $AB$  is also given in position: Therefore ( $i$ ) the point  $B$  is given. But the point  $G$  is also given, therefore ( $k$ ) the right-line  $GB$  is given in position. But the right-line  $CD$  is also given in position: Therefore ( $l$ ) the angle  $BGD$  is given. Wherefore if  $EF$  be parallel to  $BG$ , the angle  $EFD$  ( $m$ ) shall be given, and consequently also the other angle  $BEF$ . But the right-lines  $BG$  and  $EF$  being not parallels, let them meet in the point  $H$ . Forasmuch as  $EB$  is parallel to  $FG$ , and  $EF$  is equal to  $GD$ , that is to say, to  $BG$ ; also  $FH$  ( $n$ ) shall be equal to  $GH$  (for  $EH$  and  $BH$  being cut proportionally ( $o$ ) by the parallel  $FG$ , as  $EF$  is to  $FH$ , so is  $BG$  to  $GH$ , and by permutation, as  $EF$  is to  $BG$ , so is  $FH$  to  $GH$ ;) Therefore ( $p$ ) the angle  $HFG$  is equal to the angle  $HGF$ , but the said angle  $HGF$  is given (for it is equal ( $q$ ) to the given angle  $BGD$ ;) Therefore the angle  $HFG$  is also given. But to that angle the angle  $BEF$  is equal; and therefore is given, as also the remaining angle  $EFG$ .

P R O P. XXXIV. Plate VII. Fig. 43.

*If from a given point  $E$ , there be drawn unto parallel right-lines  $AB$  and  $CD$ , given in position, a right-line  $EFG$ , that right-line  $EFG$  shall be divided in a given ratio (to wit) as  $EF$  to  $FG$ .*

*Constr.* For from the point  $E$  let there be drawn the line  $EH$ , perpendicular to the line  $CD$ .

*Demonstr.* Forasmuch as from the given point  $E$  there is drawn to the line  $CD$  the right-line  $EH$ , making the given angle  $EHG$ , ( $a$ ) the said line  $EH$  is given in position, but both the one and the other lines  $AB$  and  $CD$  is also given in position. Therefore ( $b$ ) the points of intersection  $K$  and  $H$ , are given. But the point  $E$  is also given: Therefore ( $c$ ) each line  $EK$  and  $KH$  is given. Wherefore ( $d$ ) the ratio of the said  $EK$  to  $KH$  is given. But as  $EK$  is to  $KH$ , so is  $EF$  to  $FG$ ; (for in the triangle  $GEH$  the line  $KF$  being parallel to  $HG$ , the sides  $EH$  and  $EG$  are cut proportionally;) Therefore the ratio of the said  $EF$  to  $FG$  is given.

S

OTHERWISE

h 6. def.  
i 25. prop.  
k 26. prop.

l sch. 30.  
prop.  
m 29. 1.

n 14. 5.  
o 2. 6.

p 5. 1.

q 15. 1.

a 30. prop.

b 25. prop.

c 26. prop.  
d 1. prop.

## OTHERWISE. Plate VII. Fig. 44.

*Constr.* To the parallel right-lines given in position, AB and CD, let there be drawn from the point E the right-line FEG : I say, that the ratio of GE to EF is given.

*Demonstr.* For from the point E let there be drawn to CD the perpendicular EH, and produced to the point K ; seeing therefore that from the point E to the right-line CD, given in position, there is drawn the line EH, making the given angle EHG, (a) the said line EH is given in position. But each line AB and CD is also given in position : Therefore (b) each point of intersection H and K is given. But the point E is also given, therefore (c) each of the lines EH and EK is given in Magnitude ; and therefore (d) the ratio of the said EH to EK is given. But (e) as EH is to FK so is EG to EF (for the opposite angles at the point E being equal, and the lines AB and CD parallels, the triangles EHG and EKF are equiangled ; and therefore as EH is to EG so is EK to EF ; and by permutation as EH to EK, so is EG to EF.) Therefore the ratio of the said lines EG to EF is given.

a 30. prop.

b 25. prop.

c 26. prop.

d 1. prop.

e 4. 6.

## PROP. XXXV. Fig. 45.

If from a given point A, to a right-line BC, given in position, there be drawn a right-line AD, which let be divided in E, in a given ratio (to wit) as AE to ED, and that by the point of section E there be drawn a right-line FEG, opposite to the right-line BC, given in position, the line FG so drawn shall be given in position.

*Constr.* For from the point A, let there be drawn the line AH, perpendicular to the line BC.

*Demonstr.* For seeing that from the given point A there is drawn to BC given in position, the right-line AH making the given angle AHD, (a) the said line AH is given in position. But BC is also given in position. Therefore (b) the point H is given. But the point A is also given : Therefore (c) the line AH is given in magnitude and in position. And seeing that (d) as AE to ED, so is AK to KH, and that the ratio of AE to ED is given, also the ratio of AK to KH is given and by compounding, (e) the ratio of AH to AK is given. But AH is given in Magnitude : Therefore (f) all

a 30. prop.

b 25. prop.

c 26. prop.

d 2. 6.

e 6. prop.

f 2. prop.



AK is given in Magnitude. But AK is also given in position, and the point A is given: Therefore (g) the point K is also given, and seeing that by the said given point K there is drawn the line FG, opposite to the right-line BC given in position; the said line FG (h) is given in position.

g 27. prop.

h 28. prop.

P R O P. XXXVI. Plate VII. Fig. 46.

If from a given point A, there be drawn to a right-line BC given in position, a right-line AD, and to it be added a right-line AE, having to the same AD a given ratio, and that through the extremity E of the added line AB, there be drawn a right-line FEK, opposite to the line BC, given in position, that same line FEK shall be given in position.

Constr. For from the point A let there be drawn to the line BC, the perpendicular AL, and let it be prolonged to the point G.

Demonstr. Forasmuch as from the given point A, there is drawn to the right-line BC, given in position, the right-line GL, which makes the given angle GLD, (a) that line GL is given in position. But BC is also given in position, therefore (b) the point L is given; and seeing that the point A is also given, the line (c) AL is given. But forasmuch as the ratio of AE to AD, is given, and that (d) as the said AE is to AD, so is AG to AL; (because the triangles ALD and AGE are equiangled) the ratio of AG to AL is also given. But AL is given in Magnitude: Therefore (e) AG is given in Magnitude. But it is also given in position, and the point A is given: Therefore (f) the point G is also given. And seeing that by the same given point G there is drawn the line FK, opposite to the right-line BC, given in position; (g) the said line FK is given in position.

a 30. prop.

b 26. prop.

c 26. prop.

d 4. 6.

e 2. prop.

f 27. prop.

g 28. prop.

P R O P. XXXVII. Fig. 47.

If unto parallel right-lines AB and CD, given in position, there be drawn a right-line EF, divided in the point G, in a given ratio, (to wit,) of EG to GF: and if through the point of section G, there be drawn opposite to the right-lines AB or CD, given in position, a right-line HGK, that line shall be given in position.

S 2

Constr.

*Constr.* For let there be taken in the line AB the given point L, and from that point let there be drawn the line LN, perpendicular to CD.

*Demonstr.* Seeing that from the given point L, there is drawn to the right-line CD, the line LN, making the given angle LND, the said LN (a) is given in position. But CD is also given in position: Therefore the point N (b) is given. But the point L is also given: Therefore (c) the line LN is given; and seeing that the ratio of FG to GE is given, and that \* as FG is to GE, so is NM to ML, the ratio of the said NM to ML is given; and by compounding, (d) the ratio of LN to LM is also given. But LN is given in Magnitude; therefore ML is (e) given in Magnitude. But it is also given in position, and the point L is given; Therefore the point M (f) is also given. And considering that through the said point M there is drawn the right-line KH, opposite to the right-line CD, given in position, the said line KH is also given in position.

Scholium.

\* EUCLIDE supposeth here, that as FG is to GE, so NM is to ML; but by another it is thus demonstrated.

The lines EF and LN are parallels or not parallels: Let them in the first place be parallels, and forasmuch as by Construction the lines EL, FN, EF, and LN, are parallels, EN shall be a parallelogram, and therefore the side EF is equal to the side LN. Again, seeing that MG is parallel to NF, and GF to MN, GN shall be also a parallelogram; and therefore the side GF is equal to the side MN. Wherefore the equal sides EF and LN, shall have to the equal sides FG and MN, (g) one and the same ratio. Therefore as EF is to FG, so is LN to MN; and by dividing, (h) as GE to GF, so is LM to MN.

Now suppose that the lines EF and LN (Fig. 48.) are not parallels, but that they meet in the point O. Forasmuch as in the triangle OFN there is drawn HK, parallel to FN one of the sides; (i) the sides OF and ON are divided proportionally; and therefore as FG is to GO, so is NM to MO. Again, seeing that in the triangle OGM there is drawn EL, parallel to the side GM, the sides OG and OM are divided proportionally: Wherefore (k) as OE is to EG, so is OL to LM, and by compounding, (l) as OG is to EG, so is OM to LM; but it hath been demonstrated that as FG is to GO, so is NM to MO; therefore in ratio of equality, (m) as FG is to GE, so is NM to ML.

PROP.

P R O P. XXXVIII. Plate VII. Fig 49.

If unto parallel right-lines AB and CD, there be drawn a right-line EF, and that to it there be added some other right-line EG, which hath a given ratio to the same EF; and if through the extremity G of the added line EG, there be drawn a right-line HK, against the parallels given in position AB and CD, the line drawn HK shall be also given in position.

*Constr.* For let there be taken in the line AB, the given point N, and from thence let there be drawn to CD the perpendicular NM, and let it be prolonged to the point L.

*Demonstr.* Forasmuch as from the given point N there is drawn to the right-line CD, given in position, the right-line NM making a given angle NMF, the said angle NMF (*a*) is given in position. But the line CD is also given in position: Therefore (*b*) the point M is given. But the point N is also given: Therefore (*c*) the line NM is given, and therefore the ratio of EG to EF is given; and because (*d*) as EG to EF, so is LN to MN, the ratio of LN to NM is also given: But NM is given, therefore LN is (*e*) also given. But the point N is given: Therefore (*f*) the point L is also given. Seeing then that by the given point L there is drawn the right-line HK, opposite to the line AB given in position, (*g*) the said line AK is also given in position.

a 30. prop.  
b 25. prop.  
c 26. prop.

d sch. 37.  
prop.  
e 2. prop.  
f 27. prop.

g 28. prop.

P R O P. XXXIX. Fig. 50.

If all the sides of a triangle ABC are given in magnitude, the triangle is given in kind.

*Constr.* For, let there be exposed the right-line DG given in position, ending in the point D; but being infinite towards the other part G, and therein let be taken DE, equal to AB.

*Demonstr.* Now seeing the said AB is given in magnitude, DE is so also; but the same DE is also given in position, and the point D is given: Therefore (*a*) the point E is given.

a 27. prop.

Again, Let EF be put equal to BC; and seeing that BC is given in magnitude, EF shall be so also. But the said EF is in like manner given in position, and the point E is given: Therefore (*b*) the point F is given.

b 27. prop

Furthermore, Let FG be taken equal to AC. Now forasmuch as the said AC is given in magnitude, FG is



c 6. def.

d 6. def.

e 25. prop.

f 26. prop.

so also. But  $FG$  is also given in position, and the point  $F$  is given: Therefore the point  $G$  is also given. Now from the center  $E$ , with the distance  $ED$ , let there be described the circle  $DHK$ , (c) and that circle shall be given in position. Again, on the center  $F$ , and distance  $FG$ , let there be described the circle  $GLK$ . Therefore (d) the said circle  $GLK$  is given in position; and therefore (e) the point of Intersection  $K$  is given. But each of the points  $E$  and  $F$  is given: Therefore each line (f)  $EK$ ,  $EF$ , and  $FK$ , is given in position and magnitude. Therefore the triangle  $FK$  is given \* in kind; but it is equal and alike to the triangle  $ABC$ ; and therefore the triangle  $ABC$  is also given in kind.

## Scholium.

g 1. prop.

h sch. 30.  
prop.

i 15. def.

\* EUCLIDE supposeth here, that a triangle whose sides are given in magnitude and position, is given in kind; but the antient Interpreters demonstrated it in a manner thus. Forasmuch as the right-lines  $KE$  and  $EF$  are given, (g) the ratio which they have to one another is also given. Also the right-lines  $EF$  and  $FK$  being given, their ratio is also given; and in like manner, the ratio of the said  $EK$  and  $FK$  is given. Again, seeing that the same lines  $KE$  and  $EF$  are given in position, (h) the angle  $KEF$  is given in magnitude: Moreover, the right-lines  $EF$  and  $FK$  being given in position, the angle  $EKF$  is given in magnitude, as is also the residue  $EKF$ , and so in the triangle  $EKF$  are all the angles given, and also the ratios of the sides: Therefore (i) the said triangles  $EKF$  is given in kind.

## P R O P. XL. Plate VII. Fig. 37, 38.

If the angles of a triangle  $ABC$ , are given in magnitude, the triangle is given in kind.

Constr. Let there be exposed the right-line  $DE$ , given in position and in magnitude; and let there be constituted at the point  $D$  the angle  $EDF$ , equal to the angle  $CBA$ , and at the point  $E$  the angle  $DEF$ , equal to the angle  $BCA$ ; therefore the third angle  $BAC$  is equal to the third angle  $DPE$ .

Demonstr.

*Demonstr.* For each of the angles constituted in the points A, B, and C, is given: Therefore each of those which are posited in the points D, F, and E, is also given; and seeing that to the right-line DE given in position, and to the point D given therein, there is drawn the right-line DF, which makes the given angle EDF, (a) the line DF is given in position; and for the same reason the line EF is given in position: Therefore (b) the point F is given in position. But each of the points D and E is given: Therefore (c) each of the lines DF, DE, and FE is given in magnitude. Wherefore the triangle DFE is given in kind; and is alike to the triangle ABC: Therefore the triangle ABC is given in kind.

a 29. prop.

b 25. prop.

c 26. prop.

P R O P. XLI. Plate VII. Fig. 37, 38.

*If a triangle ABC, bath one angle BAC given, and that the two sides BA and AC, which constitute it, have to one another a given ratio, the triangle is given in kind.*

*Constr.* For, let there be exposed the right-line DF given in magnitude and position. And thereon, and at the given point F, let there be constituted the angle DFE equal to the angle BAC.

*Demonstr.* Now the angle BAC is given: Therefore also the angle DFE is given, and seeing that to the right-line DF given in position, and from the given point F therein is drawn a right-line FE, making the given angle DFE, (a) the said line FE is given in position. But seeing that the ratio of AB to AC is given, let the same be made of DF to FE, then let DE be drawn. Therefore the ratio of DF to FE is given. But DF is given: Therefore (b) FE is given in magnitude. But the same FE is also given in position, and the point F is given. Therefore (c) the point E is also given. But each of the points D and F is given: Therefore (d) each of the right-lines DF, FE, and DE is given in position and magnitude. Wherefore (e) the triangle DEF is given in kind. And seeing that the two triangles ABC and DEF have an angle equal to an angle, that is to say, the angle BAC equal to the angle DFE, and the sides which constitute those equal angles, proportional; (f) the triangle ABC is alike to the triangle DEF. But the triangle DEF is given in kind: Therefore the triangle ABC is given in kind.

a 29. prop.

b 2. prop.

c 27. prop.

d 26. prop.

e 39. prop.

f 6. 6.

## PROP. XLII. Plate VIII. Fig. 1, 2.

*If the sides of a triangle ABC, are to those of another in given ratio's, the triangle ABC is given in kind.*

*Constr.* For, let there be exposed the right-line D, given in magnitude, and seeing that the ratio of BC to AC is given, let the same be made of D to E.

a 2. prop.

b 2. prop.

c 39. prop.

d 5. 6.

*Demonstr.* Now D is given, therefore (a) E is also given. Again, seeing that the ratio of AC to AB is given, let the same be made of E to F. Now E is given, therefore (b) F is also given. Now of three right-lines, equal to the three given right-lines D, E, and F, (and of which three lines, two of them, in what manner soever they be taken, are greater than the other,) let there be constituted the triangle GHK, in such sort as D may be equal to HK; but E is equal to KG, and GH, equal to F; therefore each of the said lines HK, KG and GH, is given in magnitude: Wherefore (c) the triangle HGK is given in kind. And seeing that as BC is to CA, so is D to E, and that D is equal to HK, and E to KG, as BC is to CA, so HK is to KG. Again, seeing that as CA is to AB, so is E to F, and that E is equal to KG, and F to GH; as CA is to AB, so is KG to GH. But it hath been demonstrated, that as BC is to CA, so is HK to KG: Therefore in ratio of equality, as BC is to AB, so is HK to GH. Therefore (d) the triangle ABC is also given in kind.

## PROP. XLIII. Fig. 3.

*If the sides BC and BA, about one of the acute angles of a rectangled triangle ABC, have to one another a given ratio, that triangle is given in kind.*

a 6. def.

*Constr.* Let there be exposed the right-line DE given in magnitude and position, and on it let there be described the semicircle DGE: Therefore (a) the semicircle DGE is given in position.

b 2. prop.

c 19. 1.

d 14. 5.

*Demonstr.* For the line DE being given, and divided in two equal parts, the center of the said circle is given in position, and the semidiameter in magnitude. And forasmuch as the ratio of BC to BA is given, let the same be made of DE to F: Therefore the ratio of DE to F is given. But DE is given, therefore F (b) is also given. Now BC is greater than (c) AB: Therefore ED is (d) also greater than F. Let DG be fitted equal to F, and let EG be drawn; then on the center D, with the

distance



distance DG, let the circle GK be described. Now that circle (*e*) is given in position, seeing that the center D is given, and the semidiameter DG is also given in magnitude. But the semicircle DGE is also given in position: Therefore (*f*) the point of intersection G is given. But the points D and E are also given, therefore (*g*) each of the right-lines DE, DG, and EG is given in position and magnitude. Wherefore (*h*) the triangle DGE is given in kind. And seeing that the triangles ABC and DGE have an angle equal to an angle, to wit, the right angle BAC to the right-angle (*i*) DGE, and the sides about the angles CBA and EDG proportional. But each of the others ACB and DEG are less than a right-angle: Therefore those triangles ABC and DEG (*k*) are alike. But the triangle DGE is given in kind: Therefore the triangle ABC is also given in kind.

e 6. def.

f 25. prop.

g 26. prop.

h 39. prop.

i 31. 3.

k 7. 6.

P R O P. XLIV. Plate VIII. Fig. 4.

If a triangle ABC, hath one angle B given, and that the sides BA and AC, about another angle BAC, have to one another a given ratio, the triangle ABC is given in kind.

*Constr.* Now the given angle B is either acute or obtuse, (for it was a right-angle in the foregoing proposition.) Let it be in the first place acute, and from the point A let AD be drawn perpendicular to BC.

*Demonstr.* Therefore the angle ADB is given: But the angle B is also given; and therefore the third angle BAD is given: Wherefore (*a*) the triangle ABD is given in kind; and therefore (*b*) the ratio of BA to AD is given. But the ratio of the same BA to AC is also given: Therefore (*c*) the ratio of AD to AC is given, and the angle ADC is a right-angle: Wherefore the triangle (*d*) ACD is given in kind: Therefore (*e*) the angle C is given. But the angle B is also given; and therefore the other angle BAC is given: Therefore (*f*) the triangle ABC is given in kind.

a 40. prop.

b 3. def.

c 8 prop.

d 43. prop

e 3. def.

f 40. prop.

*Constr.* Now let the angle ABC be obtuse, and on the side CB prolonged, let there be drawn the perpendicular AD.

*Demonstr.* Forasmuch as the angle ABC (*Fig. 5.*) is given, the angle ABD, which follows it, shall be given. But the angle ADB is also given: Therefore the third angle DAB is given. Wherefore (*g*) the triangle ABD is given in kind; and therefore (*h*) the ratio of DA to AB is given. But the ratio of AB to AC is also given:

g 40. prop.

h 3. def.

given:

i 8. *prop.*

given: Therefore (i) the ratio of DA to AC is given, and the angle D is a right-angle. Therefore the triangle DAC is given in kind, and therefore the angle ACB is given. But the angle ABC is also given: Therefore the third angle BAC is given. Wherefore the triangle ABC is given in kind.

## P R O P. XLV. Plate VIII. Fig. 4.

*If a triangle ABC hath one angle BAC given, and that the line compounded of the two sides AB and AC, about the said given angle BAC, hath to the other side BC a given ratio, the triangle ABC is given in kind.*

a 7. *prop.*

*Constr.* For, let the angle BAC be divided into two equal parts by the line AD, therefore (a) the angle CAD is given.

b 3. 6.

c 18. 5.

*Demonstr.* Seeing that as AB is to AC, so (b) is BD to CD; by compounding, (c) as the line compounded of CAB is to CA, so is BC to CD, and by permutation, as the line compounded of CAB is to CB, so is CA to CD. But the ratio of the line compounded of CAB to BC, is given; therefore the ratio of CA to CD is also given, and the angle CAD is given. Therefore (d) the triangle ACD is given in kind, and therefore the angle C is given. But the angle BAC is also given: Therefore the third angle B is given: Wherefore (e) the triangle ABC is given in kind.

d 44. *prop.*e 40. *prop.*

## O T H E R W I S E. Fig. 6.

*Constr.* Let BA be prolonged directly unto the point D, so that AD may be equal to AC, and let CD be joined.

f 12. 1.

g 5. 1.

h 44. *prop.*i 40. *prop.*

*Demonstr.* Forasmuch as the ratio of the line compounded of CAB to CB is given, and that AD is equal to AC, the ratio of the whole line BD to BC is given. But the angle ADC is also given, for it is the half of the given angle BAC (for the said angle BAC (f) is equal to the two internal angles ACD and ADC, which are (g) equal to one another, because the sides AC and AD are equal:) Wherefore the triangle BDC (h) is given in kind, and therefore the angle B is given. But the angle BAC is also given. Therefore the remaining angle ACB is given: Wherefore (i) the triangle ABC is given in kind.

P R O P.

R P O P. XLVI. Plate VIII. Fig. 4.

If a triangle ABC hath one angle B given, and that the line CAB compounded of the two sides AC and AB about another angle BAC hath to the other side BC a given ratio, the triangle ABC is given in kind.

*Constr.* For let the angle BAC be divided into two equal parts, by the line AD.

*Demonstr.* Therefore (as hath been shewn in the foregoing proposition) the compound line CAB is to CB, as AB is to BD. But the ratio of the said compound line CAB to CB is given: Therefore also the ratio of AB to BD is given. But the angle B is also given: Therefore the triangle ABD (a) is given in kind; and therefore (b) the angle BAD is given. But the angle BAC is double to that of BAD; and therefore it is also given. Therefore the third angle C is given. Wherefore the triangle ABC is given in kind.

a 41. prop.  
b 2 def.

OTHERWISE. Fig. 6.

*Constr.* Let BA be prolonged directly, and let AD be put equal to AC, and let CD be joined.

*Demonstr.* Forasmuch as the ratio of the line compounded of CAB to CB is given, and that AD is equal to AC, the ratio of BD to BC is given; and the angle B is also given: Therefore the triangle CDB (c) is given in kind; and therefore (d) the angle D is given: Therefore the angle BAC which is double to BDC, is also given: Wherefore the other angle ACB is given; and therefore the triangle ABC is given in kind.

c 41. prop.  
d 3 def.

P R O P. XLVII. Fig. 7.

*Rectiline figures, AB, CDE, given in kind, are divided into triangles given in kind.*

*Constr.* For let the right-lines EB and EC be drawn.

*Demonstr.*



a 3. def.

b 41. prop.

c 4. prop.

d 8. prop.

e 41. prop.

*Demonstr.* Forasmuch as the rectiline figure ABCDE is given in kind, the angle (a) BAE is given, and the ratio of the side AB to AE is also given: Therefore (b) the triangle BAE is given in kind. Wherefore the angle ABE is given. But the whole angle ABC is also given: Therefore (c) the remaining angle EBC is given. But the ratio of the side AB to the side BE, and also that of AB to BC is given: Therefore (d) the ratio of BC to BE is given, and the angle CBE is also given: Therefore (e) the triangle BCE is given in kind. By the same way it may be demonstrated that the triangle CDE is given in kind. Therefore rectiline figures given in kind divide themselves into triangles given in kind.

## P R O P. XLVIII. Plate VIII. Fig. 8.

*If on one and the same right-line AB, are described triangles, as ACB and ABD, given in kind, those triangles shall have to one another a given ratio, as ACB to ABD.*

*Constr.* For from the points A and B, let there be drawn at right-angles on the line AB, the lines AE and BG, and prolonged unto the points F and H; through the points C and D, let there be drawn the lines ECG and FDH, parallel to AB.

a 3. def.

b 40. prop.

c 80. prop.

d 3. def.

e 8. prop.

f 1. 6.

g 41. 1.

h 15. 5.

*Demonstr.* Forasmuch then as the triangle ABC is given in kind, (a) the ratio of CA to BA is given, and the angle CAB is also given; but the angle BAE is given: Therefore the remaining angle CAE is also given; but the angle CAE is given; and therefore the other angle ACE is also given. Wherefore (b) the triangle AEC is given in kind. Now the ratio of EA to AB (c) is given; (for (d) the ratio of EA to AC, and that of AC to AB is given;) and in like manner, the ratio of FA to AB is given. Therefore (e) the ratio of EA to AF is given; but as AE is to AF, so is (f) the parallelogram AH to the parallelogram AG; but ACB is (g) the half of AH, and ADB the half of AG; therefore the ratio of the triangle ACB to the triangle of ADB is given; for it is the same ratio with that of AH to AG (h); that is to say, of EA to AF, which is given.

P R O P

P R O P. XLIX. Plate VIII. Fig. 9.

If on one and the same right-line AB, there are described any two rectiline figures AECFD and ADB, given in kind, they shall have to one another a given ratio (to wit) AECFB to ADB.

Constr. For let the lines FA and FE be drawn : Therefore each of the triangles (a) ABF, AFE, and ECF is given in kind.

Demonstr. Seeing that on one and the same right-line EF there are described the triangles ECF and EAF, given in kind ; the ratio of ECF to EAF (b) is given. Therefore by compounding, (c) the ratio of AECF to EAF is given. But the ratio of the said EAF to FAB is given, (d) because they are triangles given in kind, described on one and the same right-line AF : Therefore (e) the ratio of AECF to FAB is given. Wherefore by compounding, (f) the ratio of AECFB to FAB is given. But the ratio of the same FAB to ABD (g) is given : Therefore (h) the ratio of AECFD to ABD is also given.

a 47. prop.

b 48. prop.

c 6. prop.

d 48. prop.

e 8. prop.

f 6. prop.

g 48. prop.

h 8. prop.

P R O P. L. Fig. 10.

If two right-lines AB and CD, have to one another a given ratio, and that on those lines there be described rectiline figures AEB and CFD, alike, and alike posited, they will have to one another a given ratio.

Demonstr. To the two lines AB and CD, let there be taken a third proportional G. Therefore as AB is to CD, so is CD to G. But the ratio of AB to CD is given : Therefore the ratio of CD to G is also given ; Wherefore (a) the ratio of AB to G is given. But (b) as AB is to G, so is AEB to CFD : Therefore the ratio of the same AEB to CFD is given.

a 8. prop.

b cor. 19.

20. 6.

P R O P. LI. Fig. 11, 12.

If two right-lines AB and CD have to one another a given ratio, and that upon them there be described any rectiline figures AEB and CFD, given in kind, they will have to one another a given ratio, (to wit, that of AEB to CFD.)

Constr.

*Constr.* For on AB let the rectangled figure AH be described alike, and alike posited to DF.

a 49. prop.

b 50. prop.

c 8. prop.

*Demonstr.* Now DF is given in kind: Therefore also AH is given in kind. But AEB is also given in kind, and described on the same line AB: Therefore (a) the ratio of AEB to AH is given: And seeing that the ratio of AB to CD is given, and that on those lines are described the rectiline figures AH and DF alike, and alike posited, the ratio (b) of the said line AH to DF is given. But the ratio of AEB to AH is also given: Therefore the ratio (c) of AEB to DF is given.

P R O P. LII. Plate VIII. Fig. 7

*If on a right-line BE, given in magnitude, there be described a figure BAE, given in kind, that figure BAE is given in magnitude.*

*Constr.* For on the same line BE let the square BD be described. Therefore BD is given in kind \* and in magnitude.

a 29. prop.

b 2. prop.

*Demonstr.* Seeing that on the right-line BE, are described the two rectiline figures DAE and BD, given in kind, (a) the ratio of BAE to BD is given: Therefore (b) BAE is given in magnitude.

Scholium.

\* The antient interpreter hath noted here that every square is given in kind; for that all the angles thereof are given; being all equal and right-angles: But also the ratio's of the sides are given; for those sides being all equal, their ratio's are also equal. Moreover, whensoever a square is exposed, a square equal thereto may be exhibited; and therefore the square is given in magnitude, as also each side thereof.

P R O P. LIII. Fig. 13, 14.

*If there are two figures AD and EH, given in kind, and that one side BD of the one, hath to a side FH of the other, a given ratio; the other sides shall have also to the other sides given ratio's.*

I

*Demonstr.*



*Demonstr.* For seeing that the ratio of BD to FH is given, and also that (a) of BD to BA, (b) the ratio of the said AB to FH is given. But the ratio of the same FH to EF (c) is also given: Therefore (d) the ratio of AB to EF is given. In like manner also the ratio's of the other sides to the other sides are given.

a 3. def.  
b 8. prop.  
c 3. def.  
d 8. prop.

P R O P. LIV. Plate VIII. Fig. 15, 16.

*If two figures A and B given in kind, have to one another a given ratio, also their sides shall be to one another in a given ratio*

*Constr.* For either the figure A is alike and alike posited to B, or is not: Let it in the first place be alike, and alike posited; and let there be taken the line G, a third proportional to the lines CD and EF.

*Demonstr.* As CD is to G, (a) so is A to B. But the ratio of A to B is given; therefore also the ratio of CD to G is given. And seeing that CD, EF, and G, are proportional, (b) also the ratio of CD to EF is given. But A and B are given in kind: Therefore (c) the other sides shall have given ratio's to the other sides.

a cor. 19.  
20. 6.  
b 24. prop.  
c 53. prop.

Now let the figure A be not alike to the figure B, and let there be described on EF the figure EH, alike and alike posited to A: Therefore the figure EH is given in kind; but the figure B is also given in kind: Therefore (d) the ratio of B to EH is given; and therefore the ratio of A to the same EH (e) is also given; But A is alike to EH: Therefore (by what is abovesaid) the ratio of CD to EF is given; and in like manner the ratio of the other sides to the other sides is given.

d 49. prop.  
e 8. prop.

O T H E R W I S E. Fig. 17.

*Constr.* Let there be exposed the given line GH: Now either the figure A is alike to the figure B, or not. Let it in the first place be alike, and let it be as CD is to EF, so is GH to LK; then on GH and LK let the figures M and N be described alike, and alike posited to the said A and B, which figures M and N shall consequently be given in kind.

*Demonstr.*

f 22. 6.

g 52. prop.

h sch. 52.  
prop.i sch. 52.  
prop.k 3. prop.  
l 53 prop.

*Demonstr.* Therefore seeing that as CD is to EF, so is GH to LK, and that on those lines CD, EF, GH, and LK, are described the figures A, B, M, and N, alike and alike posited; (f) as A is to B, so is M to N. But the ratio of A to B is given: Therefore the ratio of M to N is given. But (g) M is given, considering that it is a figure given in kind, described on a right-line given in magnitude; therefore N is also given.

*Constr.* 2. Now, on LK let the square O be described: Therefore (h) the figure O is given in kind.

*Demonstr.* 2. Wherefore the ratio of O to N is given. But N is given: Therefore O is given; and consequently (i) also KL. But GH is given: Therefore (k) the ratio of GH to KL is given. But as GH is to LK, so is CD to EF. Therefore the ratio of CD to EF is given; and therefore the figures A and B being given in kind, (l) the other sides of the same figures shall also have to the other sides given ratio's. But if the figures be not alike, the latter part of the demonstration here above must be observed.

## P R O P. LV. Plate VIII. Fig. 18.

*If a space A be given in kind, and in magnitude, the side thereof shall be given in magnitude.*

*Constr.* For let the right-line BC, given in position and in magnitude, be exposed; and thereon let there be described the space D, alike and alike posited to A; therefore the said space D is given in kind.

*Demonstr.* For as it is described on the line BC, given in magnitude, it is also (a) given in magnitude. But the figure A is also given: Therefore (b) the ratio of A to D is given. But those figures A and D are given in kind: Therefore (c) the ratio of the line EF to the line BC is given. But BC is given: Therefore (d) EF is also given. But the ratio of the same EF to FG is given: Therefore (e) FG is given. And by the same ways of reasoning it may be demonstrated that each of the other sides are given in magnitude.

a 52. prop.

b 3. prop.

c 54. prop.

d 3. def.

e 2. prop.

## O T H E R W I S E. Fig. 19.

*Constr.* Let the space GHIKL be given in kind and in magnitude: I say that the sides thereof are given in magnitude. For on the right-line GH let there be described the square GM; therefore (f) GM is given in kind.

*Demonstr.*

*Demonstr.* But the space GHIKL is also given in kind : Therefore (g) the ratio of the same space GK to GM is given. But GK is given in magnitude : Therefore (b) GM is also given in magnitude ; and seeing that GM is the square of the line GH, (i) that line GH is given in magnitude. Wherefore in like manner, each of the other lines HI, IK, KL, and LG, is given.

g 49. prop.  
h 2. prop.

i sch. 52.  
prop.

PROP. LVI. Plate VIII. Fig. 20.

*If two equiangled parallelograms A and B, have to one another a given ratio, as one side CD of the first A, is to one side FG, of the second B ; so the other side GE, of the second B, is to that to which DH, the other side of the first A, hath the given ratio that the parallelogram A hath to the parallelogram B.*

*Constr.* For let HD be prolonged directly to L, so that as CD is to FG, so HD may be to DL ; and finish the parallelogram DK.

*Demonstr.* Seeing that as CD is to FG, so HD is to DL, and (a) that CD is equal to KL ; as LK is to FG, so is GE to DL ; and thus the sides about the equal angles DLK and EGF are reciprocally proportional : Wherefore (b) DK is equal to B ; and therefore seeing the ratio of A to B is given, and that B is equal to DK, the ratio of A to DK is given. But as (c) A is DK (that is to B) so is HD to DL : therefore the ratio of HD to DL is also given : and seeing that as CD is to FG, so GE is to DL, and that the right-line HD hath to DL a given ratio ; to wit, that which the space A hath to the space B ; as CD is to FG, so GE is to that to which HD hath the given ratio that the space A hath to the space B, that is to say, the ratio of HD to DL.

a 34. i.

b 14. 6.

c i. 6.

PROP. LVII. Fig. 21.

*If a given space AD be applied to a given right-line AB in a given angle CAB, the breadth CA of the application is given.*

*Constr.* For on AB, let there be described the square AF ; therefore (a) the same AF is given : Let the lines EA, FB, and CD, be prolonged to the points G and H.

a sch. 52.  
prop.

*Demonstr.* Seeing therefore that each space AD and AF is given, their ratio is also given. But (b) AD is equal to AH : Therefore the ratio of AF to AH is given : Wherefore the ratio of EA to AG is given. (For (c) it

b 36. i.  
c i. 6.



d 40. prop.

is the same with that of AF to AH.) But EA is equal to AB; therefore the ratio of AB to AG is given. Now seeing that the angle CAB is given, and the angle GAB also given, the residue CAG is given. But the angle CGA is also given, being a right-angle: Therefore the remaining angle ACG is given. Wherefore the triangle (d) CAG is given in kind: Therefore the ratio of CA to AG is given. But the ratio of AB to the same AG is also given: Therefore the ratio of CA to AB is given; and the said AB is given: Wherefore CA is also given.

## PROP. LVIII. Plate VIII. Fig. 22.

*If a given space AB, be applied to a given right-line AC, wanting by a figure DE, given in kind, the breadths of the defects are given.*

*Constr.* For let AC be divided in two equal parts in the point F: Therefore as well AF as FC is given. On the said line FC let there be described the rectangled figure FG alike and alike posited to DE. Therefore FG is given in kind.

a 52. prop.

b 36. 1.

c 43. 1.

d 4. prop.

e 24. 6.

f 55. prop.

g 34. 1.

h 4. prop.

i 3. def.

k 2. prop.

*Demonstr.* Seeing the figure FG is described on the right-line FC given in magnitude, the said rectiline FG is (a) also given in magnitude. But FG is equal to AB and IL; (for (b) AI and FE being equal, and (c) FB and BG also equal, the Gnomon ICL is equal to AB; and therefore their added figure IL common to both, FG shall be equal to AB and IL:) Therefore the figures AB and IL together are given in magnitude. But AB is given in magnitude: Therefore (d) the remaining figure IL is also given in magnitude. But it is also given in kind, seeing it is (e) alike to DE: Therefore (f) the sides of the same IL are given: Wherefore IB is given; and seeing that it is equal (g) to FD, the same FD is also given. But FC is given; therefore the remainder DC (h) is given; and (i) in a given ratio to BD, and therefore (k) BD is given.

## PROP. LIX. Fig. 23.

*If a given space AB be applied according to a given right-line AC, exceeding it by a figure CB given in kind, the breadths of the excesses CE and CF are given.*

*Constr.* For DE being divided into two equal parts in G, let there be described on GE the rectiline figure GH, alike and alike posited to CB.

*Demonstr.*

*Demonstr.* Now seeing that CB is alike to GH, those figures CB and GH \* are about one and the same diameter, and GH is given in kind, as is CB. But it is described on the given line GE: Therefore (a) the same GH is also given in magnitude. But AB is given: Therefore AB and GH are given in magnitude. Now those figures AB and GH, are equal to LI, (for AG, LE, and EI, being equal, the Gnomon GFH is equal to AB; and therefore adding GH common to both, LI shall be equal to AB and GH;) therefore LI is given in magnitude; but is also given in kind, since it is (b) alike to CB. Therefore (c) the sides of the said LI are given, seeing it is equal to GE: Therefore (d) the remainder CF is given, and in a given ratio (e) to CE. Wherefore (f) CE is given.

a 32. prop.

b 24. 6.

c 55. prop.

d 4. prop.

e 3. def.

f 2. prop.

Scholium. Plate VIII. Fig. 24.

\* EUCLIDE supposeth here, that CB and GH are about one and the same diameter, but we shall thus demonstrate it: Let CB and GH be two alike parallelograms disposed as above, that is to say, that the equal angles join together in E, the side CE meets directly with his homologous side EH, and the side BE, his correspondent side EG; and let the diameter FE be drawn, I say that the said diameter FE prolonged, will pass through the point K; that is to say, the parallelograms GH and CB, consist about one and the same diameter. For if it be denied, the diameter EF being produced, will pass above the point K, or below it. Let it in the first place pass above it, and let it cut GK, prolonged in the point M, and through the point M let there be drawn MN, parallel to KH, which shall meet EH, prolonged in the point N, and FB in O.

*Demonstr.* Forasmuch as the parallelograms GN and CB are with the parallelogram LO about one and the same diameter, they are (g) alike to one another. Wherefore as FC is to CE, so is EG to GM. In like manner, seeing the parallelograms CB and GH are alike, as FC is to CE, so is EG to GK: Therefore (h) as EG is to GM, so is EG to GK. Wherefore (i) GM and GK are equal, a part to the whole, which is absurd: By the same way of reasoning it may be demonstrated. that the diameter prolonged will not fall below the point K: Therefore the parallelograms CB and GE consist about one and the same diameter.

g 24. 6.

h 11. 5.

i 9. 5.



## PROP. LX. Plate VIII. Fig. 25.

If a parallelogram AB, given in kind and in magnitude, be augmented or diminished by a Gnomon CFD, the breadth of the Gnomon (consisting of the lines CE and DG) are given.

a 55. prop.

*Demonstr.* For seeing that AB is given, and the Gnomon CFD also given, the whole parallelogram BF is given: But it is also given in kind, seeing it is alike to BA: Therefore (a) the sides of the same BF are given; and therefore each of the lines BE and BG is given. But each of the lines BC and BD is given; therefore each of the remaining lines CE and DG is also given.

*Constr.* Now let the parallelogram BF, given in kind and in magnitude, be diminished by the given Gnomon CFD: I say that each of the lines CE and DG is given.

b 55. prop

*Demonstr.* For seeing that BF is given, and the Gnomon CFD given, the remaining figure AB is also given. But it is also given in kind, seeing it is alike to BF: Therefore (b) the sides of the said AB are given, and therefore each of the lines CB and BD is given, but each of the lines BE and BG is given: Therefore also each of the remaining lines CE and DG is given.

## PROP. LXI. Fig. 26.

If to one side of a figure ABCE, given in kind, there be applied a parallelogrammic space CD in a given angle BCF, and that the given figure AC hath to the parallelogram CD a given ratio, the parallelogram CD is given in kind.

*Constr.* For through the point B, let BH be drawn parallel to CE, and through the point E let EH be drawn parallel to CB, and let EC and HB be prolonged to the points K and G.

a 3. def.

b 49. prop.

c 36. 1.

d 8. prop.

e 1. 6.

f 8 prop.

*Demonstr.* Forasmuch as the angle BCE is given, and the ratio of EC to CB, (a) the parallelogram CH is given \* in kind. But the figure ABCE is also given in kind, and is described on the same line BC, as the parallelogram CH given in kind is: Therefore (b) the ratio of the figure ABCE to the parallelograms CH is given. But by supposition the ratio of the said figure ABCE to the parallelogram CD is also given; and CD is (c) equal to CG: therefore (d) the ratio of CH to CG is given. Wherefore the ratio of the line EC to the line CK is given; (for (e) as CH is to CG, so is EC to CK.) But the ratio of EC to CB is also given: Therefore (f) the ratio of the said CB to CK is given. And seeing



seeing that the angle ECB is given, also the following angle BCK (*g*) is given. But the angle BCF is proposed given; and therefore the remaining angle FCK is given. Also the angle CKF is given, for that (*b*) it is equal to the angle BCK: Therefore the other angle CFK is given: Wherefore (*i*) the triangle FCK is given in kind; and therefore the ratio of FC to CK is given. But the ratio of CB to the same CK is also given. Therefore (*k*) the ratio of FC to CB is given; and the angle BCF is also given. Wherefore the parallelogram CD is given in kind.

g 13. 1 &  
4. prop.  
h 29. 1.

i 40. prop.

k 8. prop.

Scholium.

\* *Altho' it be manifest that a parallelogram that hath one angle given, and the ratio of the sides about the same angle also given, is given in kind, as Euclide declares, yet the antient interpreter thus demonstrates it.*

Seeing that in the parallelogram CH the angle ECB is given, the angle CEH is also given; for the right-line EC falling on the parallels EH and CB, doth make the two internal angles on the same part equal to two right-angles. And therefore seeing that the angle ECB is given, the other angles are given; and seeing that the ratio of EC to CB is given, and that BH is equal to CE, and EH to BC, the ratio of the sides to one another is also given.

P R O P. LXII. Plate VIII. Fig. 11, 12.

If two right-lines AB and CD, have to one another a given ratio, and that on one of them AB, there be described a figure AEB, given in kind; but on the other CD, a parallelogrammic space DF in a given angle DCF, and that the figure AEB hath to the parallelogram DF a given ratio, the parallelogram DF is given in kind.

*Constr.* For on the line AB let there be described the parallelogram AH, alike and alike posited to DF.

*Demonstr.* Seeing that the ratio of AB to CD is given, and that on those lines are described the rectiline figures AH and FD, alike and alike posited, (*a*) the ratio of AH to FD is given. But the ratio of FD to AEB is also given: Therefore (*b*) the ratio of AH to AEB is given. But the angle ABH is also given, being equal to the angle FCD, and so the figure AEB is given in kind; and to AB one of the sides thereof, the parallelogram

a 50. prop.

b 8. prop.

c 61. prop.

AH is applied in a given angle ABH, and the ratio of the said figure AEB to the said parallelogram AH is given: Therefore (c) the parallelogram AH is given in kind; and therefore FD which is alike thereto, is also given in kind.

## P R O P. LXIII. Plate VIII. Fig. 27.

If a triangle ABC be given in kind, the squares BE, CD, and CF, which is described on each of the sides, shall have a given ratio to the triangle ABC.

a 49. prop.

Demonstr. For seeing that on one and the same right-line BC, there are described the two rectiline figures ABC and CD, given in kind, (a) the ratio of the same ABC to CD is given; and therefore the ratio of the squares BE and CF, to the triangle ABC is also given.

## P R O P. LXIV. Fig. 5.

If a triangle ABC, hath an obtuse angle ABC given, that space by which the side AC subtending the obtuse angle ABC, is more in power than the sides AB and BC, that comprehend the said angle, shall have a given ratio to the triangle ABC.

a 12. 2.

Constr. Let the line CB be prolonged directly, and from the point A let the perpendicular AD be drawn: I say that the space by which the square of the line AC doth exceed the squares of the lines AB and BC, that is to say, (a) the double of the rectangle contained under CB and BD, shall have a given ratio to the triangle ABC.

b 40. prop.

c 3. def.

d 1. 6.

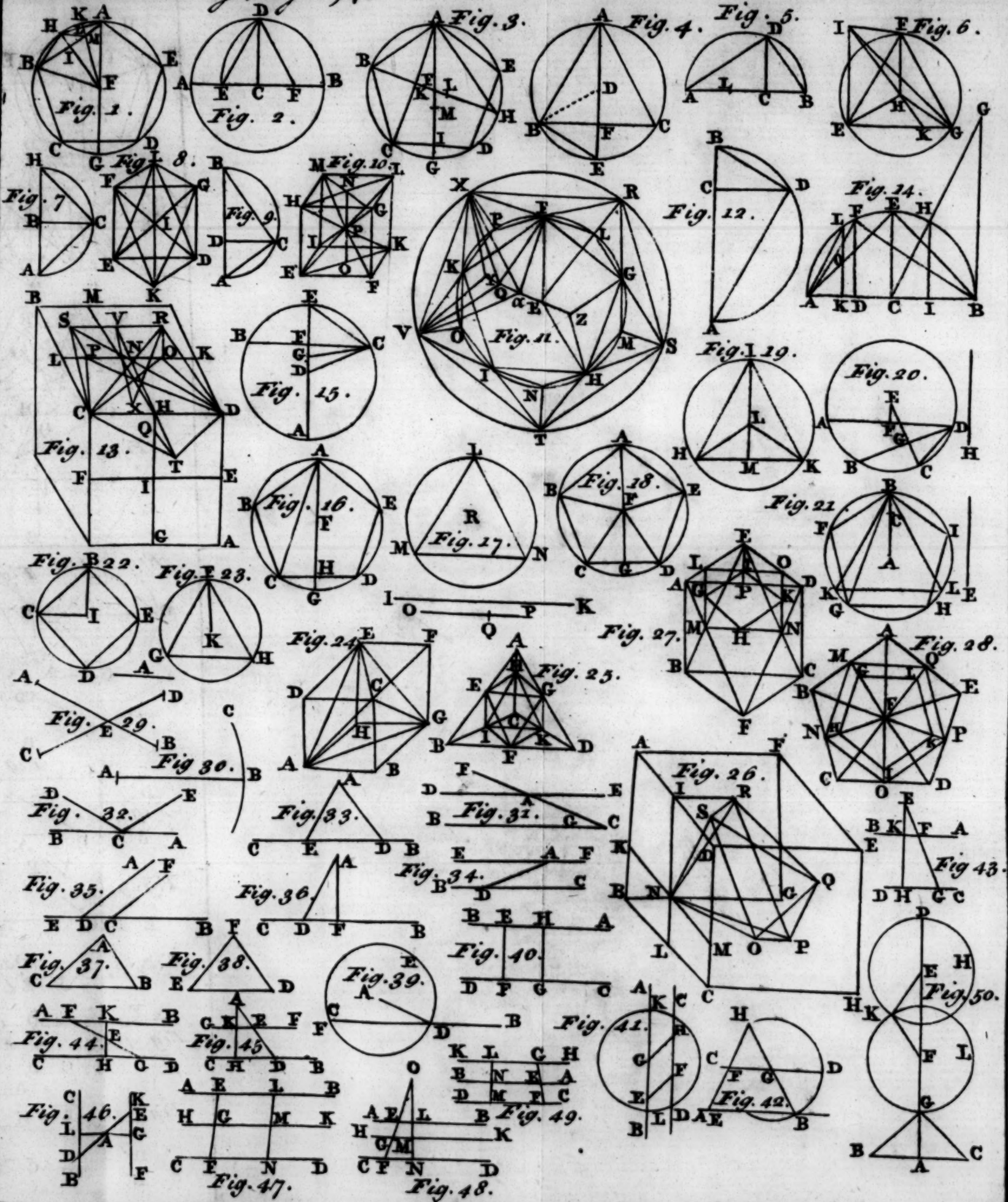
Demonstr. For seeing that the angle ABC is given, the angle ABD is also given; but the angle ADB is also given; therefore the other angle BAD is given: Wherefore (b) the triangle ABD is given in kind; therefore (c) the ratio of AD to DB is given. But as AD to DB, so (d) the rectangle of AD and BC is to the rectangle of BC and BD. But the ratio of AD to BD is given: Therefore also is the ratio of the rectangle of AD and BC to the rectangle of BC and BD given: Wherefore the ratio of the double of the said rectangle BC and BD to the rectangle of AD and BC is also given. But the said rectangle of AD and BC hath also a given ratio to the triangle ABC (to wit, a double ratio; for the rectangle is (e) double to the triangle) therefore the ratio of the double of the rectangle of BC and BD (f) to

e 41. 1.

f 8. prop.









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to the triangle ABC is given. But the same double of the rectangle of CB and BD is that space by which the square of the line AC doth exceed the squares of the lines AB and BC: Therefore the same space hath a given ratio to the triangle ABC.

P R O P. LXV. Plate VIII. Fig. 4.

*If a triangle ABC, hath one acute angle ACB given, that space, by which the side subtending the said acute angle is less in power than the sides comprehending the same acute angle, shall have a given ratio to the triangle.*

*Constr.* From the point A let there be drawn the line AD, perpendicular to BC: I say, that space by which the square of the line AB is less than the squares of the lines AC and CB, that is to say, (a) the double of the rectangle of BC and CD, hath a given ratio to the triangle ABC.

a 13. 2.

*Demonstr.* For seeing that the angle C is given, and the angle ADC also given, the other angle DAC is given: Wherefore the triangle (b) ADC is given in kind; and therefore the ratio of AD to DC is given, and consequently also (c) that of the rectangle of BC and CD to the rectangle of BC and AD: Therefore the ratio of the double of the rectangle of BC and CD to the rectangle of BC and AD is given. But the ratio of the same rectangle of BC and AD to the triangle ABC is given (for (d) the rectangle is double to the triangle:) Therefore (e) the ratio of the double of the rectangle of BC and CD to the triangle ABC is given. And seeing that the same double of the rectangle of BC and CD is that whereby the square of the line AB is less than the squares of the lines AC and BC, that space by which the square of the line AB is less than the squares of the lines AC and BC, shall have a given ratio to the triangle ABC.

b 40. prop.

c 1. 6.

d 41. 1.

e 8. prop.

P R O P. LXVI. Fig. 4.

*If a triangle ACB, hath one angle B given, the rectangle made of the lines AB and BC, containing the same angle, shall have a given ratio in the triangle.*

*Constr.* For from the point A let AD be drawn perpendicular to CB.

*Demonstr.* Therefore seeing that the angle B is given, and also the angle ADB; the other angle BAD is likewise given. Wherefore the triangle ADB (a) is given

a 40. prop.

T 4.

in



b 1. 6.

c 41. prop.

d 8. prop.

in kind; and consequently the ratio of AB to AD is given. But as AB is to AD, (b) so the rectangle of AC and CB is to the rectangle of CB and AD: Therefore the ratio of the rectangle of AC and CB to the rectangle of CB and AD is given. But the ratio of the said rectangle of CB and AD to the triangle ACB is also given; (for that it is double ratio, the rectangle being double (c) to the triangle :) Therefore (d) the ratio of the rectangle of AC and CB to the triangle ABD is given.

## P R O P. LXVII. Plate VIII. Fig 28.

*If a triangle ABC hath one angle BAC given, that space by which the square of the line compounded of the two sides BA and AC, that contain the same given angle BAC doth exceed the square of the other side, shall have a given ratio to the triangle ABC.*

*Constr.* For let BA be prolonged in such sort as that AD may be equal to AC, then having drawn DCE infinitely, from the point B let BE be drawn parallel to AC, meeting the said DE in the point E.

a 4. 6. &amp;

14. 5.

*Demonstr.* Forasmuch as AD is equal to AC, (a) DB is equal to BE; (for the two triangles ADC and BDE are alike) and from the top B is drawn to the base DE, the right-line BC: Therefore \* the rectangle of DC and CE, with the square of BC, is equal to the square of BD; but the same BD is compounded of BA and AC; therefore the square of the compound of AB and AC is greater than the square of BC, of the rectangle of DC and CE.

b 40. prop.

c 50. prop.

d 2. 6.

e 1. 6.

f 1. 6.

Now I say that the rectangle of DC and CE hath a given ratio to the triangle ABC: Forasmuch as the angle BAC is given, the angle DAC is also given. But each of the angle ADC and ACD is given, it being the half of the angles BAC which is given. Therefore (b) the triangle ADC is given in kind; and therefore the ratio of DA to DC is given. Therefore (c) the ratio of the square of the said DA to the square of DC is also given. And seeing that as BA is to AD, (d) so is EC to CD, and also as BA is to AD, (e) so is the rectangle of BA and AD to the square of AD; and as EC is to CD, (f) so also is the rectangle of EC and CD to the square of CD; by permutation, as the rectangle of BA and AD is to the rectangle of EC and CD, so is the square of AD to the square of DC. But the ratio of the said square of AD to the square of DC is given: Therefore the

the ratio of the rectangle of BA and AD to the rectangle of EC and CD is also given. But AD is equal to AC: Therefore the ratio of the rectangle of BA and AC to the rectangle of EC and CD is given. But the ratio of the rectangle of BA and AC to the triangle ABC (*g*) is given, because the angle BAC is given: Therefore (*h*) the ratio of the rectangle EC and CD to the triangle ABC is given. But the rectangle of EC and CD is that whereof the square of the line compounded of BA and AC is greater than the square of BC: Therefore that space by which the square of the line compounded of BA and AC is greater than the square of BC, shall have a given ratio to the triangle ABC.

*g* 66. *prop.*  
*h* 8. *prop.*

Scholium.

\* EUCLIDE supposeth in this place, that when in an Isosceles triangle a right-line is drawn from the top to the base, the square of that line, with the rectangle contained under the segments of the bases, is equal to the square of either of the other legs, which the antient interpreter doth thus demonstrate.

*Constr.* Let ABC be an Isosceles triangle, whose legs are AB and AC; and from the top A let AD be drawn to the base BC: I say, that the square of AD with the rectangle of BD and DC, is equal to the square of either of the legs AB or AC.

*Demonstr.* Now the line AD (*Fig. 4.*) is perpendicular to BD, or not: Let it in the first place be perpendicular: Therefore it will cut the base BC into two equal parts in the point D; and therefore the rectangle contained under BD and DC is equal to the square of the said BD, and adding to them the common square of AD, the rectangle of BD and DC with the square of AD, shall be equal to the squares DB and AD. But to those squares of AD and DB (*i*) the square AB is equal: Therefore the square of AB is equal to the rectangle of BD and DC, and the square of AD together.

*i* 47. 1.

Now suppose AD not to be perpendicular, but that from the point A there doth fall on BC the perpendicular AE, that being so, BC shall be cut into two parts equally in the point E, and unequally in D. Wherefore the rectangle of BD and DC, with the square of DE, (*k*) shall be equal to the square of BE; and adding the common square of AE, the rectangle of BD and DC, with the squares of DE and AE, shall be equal to the squares of BE

*k* 5. 2.



l 47. 1.

BE and AE. But (l) the square AD is equal to the two squares of DE and AE: Therefore the rectangle of BD and DC, with the square of AD, is equal to the squares of BE and AE. But to these squares of BE and AE the square of AB is equal: Therefore the square of AD, with the rectangle of BD and DC, is equal to the square of AB.

## O T H E R W I S E.

*Constr.* Having done, as in the foregoing Demonstration, from the point A, (Fig. 29.) let AF be drawn perpendicular to CD, and let AE be drawn.

m 8. prop.

n 1. 6.

o 41. 1.

p 37. 1.

q 8. prop.

*Demonstr.* Forasmuch as the angle BAC is given, the half thereof ACF shall be also given. But the angle AFC is given; and therefore the triangle AFC is given in kind: Therefore the ratio of AF to FC is given. But the ratio of CD to the same FC is also given, seeing that CD is double to FC: Therefore (m) the ratio of CD to AF is given; and therefore also the ratio of the rectangle of CD and EC, to the rectangle of AF and EC, is given; (for it is the same ratio (n) as that of CD to AF.) But the ratio of the rectangle of AF and EC to the triangle ACE is given; seeing it is double (o) to the same triangle. Therefore the ratio of the rectangle of CD and CE to the triangle ACE is also given. But the triangle ACE is equal to the triangle ABC (p), they being both constituted on one and the same base AC and between the same parallels AC and BE: Therefore (q) the ratio of the rectangle of CE and CD to the triangle ABC is given. But the said rectangle of CE and CD is the space by which the square of the line compounded of AB and AC, is greater than the square of BC: Therefore that space by which the square of the line compounded of AB and AC is greater than the square of BC, hath a given ratio to the triangle ABC.

## O T H E R W I S E. Fig. 4.

r 47. 1.

s 5. 2,

For the given angle A is either a right, acute, or obtuse angle: Let it in the first place be supposed a right-angle: Therefore the square of the line compounded of BAC, is greater than the square of BC, by twice the rectangle of BA and AC; (seeing that (r) the square of BC is equal to the squares of BA and AC) and the square of the line compounded of BAC (s) equal



equal to those two squares of BA and AC, and twice the rectangle of the said BA and AC :) Wherefore the ratio of double the rectangle of BA and AC to the triangle ABC is given.

*Constr.* Now let the angle C (*Fig. 4.*) be supposed acute, and from the point A let there be drawn on CB the perpendicular AD.

*Demonstr.* Forasmuch as the triangle CAB is an Oxionium triangle, and the perpendicular AD being drawn, the squares of CA and CB are equal (*r*) to the square of AB with twice the rectangle of CB and CD; adding therefore the common double rectangle of CA and CB, the squares of CA and CB, with the double rectangle of the said CA and CB, that is to say, (*u*) the square of the line compounded of ACB, are equal to the square of AB, with the double of the rectangle of CD and CB, and over and above the double of the rectangle of AC and CB, that is to say, the double of the rectangle contained under the compound-line of ACD and CB (for the rectangle of ACD and CB is (*x*) equal to the rectangles of AC and CB, and of CD and CB :) Therefore the square of the line compounded of ACB is greater than the square of AC, by double the rectangle of ACD and CB. And seeing that the angle ACB is given, and the angle BDA also given, the other angle CAD is given: Therefore (*y*) the triangle CAD is given in kind, and therefore the ratio of CD to CA is given, and by consequence the ratio of the line compounded of CD to CA (*z*) is also given. Wherefore the ratio of the rectangle of those lines compounded of ACD and CB (*a*) to the rectangle of AC and CB is also given. And the ratio of the said rectangle of AC and CB to the triangle CAB (*b*) is given, seeing the angle C is given; therefore the ratio of double the rectangle of the line compounded of ACD and CB to the triangle CAB is given.

Lastly, let the angle BAC (*Fig. 30.*) be supposed to be obtuse, and having prolonged BA, from the point C, the perpendicular CE be drawn on the said line BA prolonged; and let AF be proposed to be equal to AE.

*Demonstr.* Forasmuch as the angle BAC is obtuse, and the perpendicular CE being drawn, the squares of AB and AC, and the double of the rectangle under BA and AF, or AE, are all alike equal (*c*) to the square of BC, adding the common double rectangle of BA and AC, the squares of the said AB and AC, with the double of

+

the

t 13. 5.

u 4. 2.

x 1. 2.

y 40. prop.

z 6. prop.

a 1. 6.

b 66. prop.

c 12. 2.

d 4. 2.

e 1. 2.

f 13. 11.

g 40. prop.

h 5. prop.

i 8. prop.

k 2. 6.

l 8. prop.

m 66. prop.

the rectangle of the same AB and AC, that is to say, (d) the square of the line compounded of BAC and the double of the rectangle of BA and AF are together equal to the square of BC, with the double of the rectangle of BA and AC. Let the common double of the rectangle of BA and AF be taken away, and there will remain the square of the line compounded of BAC, equal to the square of BC, with the rectangle of AB and CF; (for the rectangle of AB and AC is equal (e) to the two rectangles of AB and AE, and of AB and CF:) Therefore the square of the line compounded of BAC is greater than the square of BC by the double of the rectangle of AB and CF. And forasmuch as the angle BAC is given, the angle CAE (f) is given. But the angle AEC is also given; therefore the other angle ACE is given: Wherefore (g) the triangle ACE is given in kind, and therefore the ratio of CA to AE, that is to say, to AF is given. Therefore (b) the ratio of the said CA to FG is also given. But the ratio of the same CA to CE is given; therefore (i) the ratio of CE to CF is also given. Wherefore the ratio of the rectangle of EC and AB to the rectangle of FC and AB is given; (for the rectangle is to the rectangle (k) as CE is to CF) and also that of the rectangle of AC and AB to the rectangle of EC and AB. Therefore (l) the ratio of the rectangle of FC and AB to the rectangle of AC and AB is given. But the ratio of the rectangle of AC and AB to the triangle ABC (m) is given: Therefore also the ratio of the double of the rectangle of FC and AB, to the triangle ABC is given. But the same double of the rectangle of FC and AB, is that whereby the square of the line compounded of BAC is greater than the square of BC, wherefore that space by which the square of the line compounded of BAC is greater than the square of BC, hath a given ratio to the triangle ABC.

## OTHERWISE. Plate VIII. Fig. 31.

*Constr.* Let the line BA be prolonged to the point D, in such sort as AD may be equal to AC, and let CD be drawn.

p 40. prop.

*Demonstr.* Forasmuch as the angle BAC is given, each of the angles ADC and ACD, which is the half thereof shall be also given; and therefore the other angle DAC is also given: Therefore (n) the triangle ACD is given in kind. Wherefore the ratio of AC to CD is given. And forasmuch as the angle ADC is given: Let each of the angle



angles DEC and AFC be made † equal to the said ADC: Therefore seeing that the angle BDC is equal to the angle DEC, and the angle DBE is common to the triangles DBE and DBC, the other angle BDE is equal to the other angle BCD; and therefore the triangle BDE is equiangular to the triangle BDC. Therefore (o) as EB is to BD, so is BD to CB: Wherefore the rectangle of EB and CB, that is to say, (p) the rectangle of EC and CB, (q) with the square of CB is equal, (r) to the square of BD, that is to say, to the square of the line compounded of BAC; for AD is equal to AC; and therefore the rectangle of EC and CB with the square of CB, that is to say, the square of the line compounded of BAC is greater than the square of the rectangle of BC and CE: I say therefore that the ratio of the said rectangle of BC and CE to the triangle ABC is given. Forasmuch as the angle BDE is equal to the angle BCD, and the angle ADC equal to the angle ACD, the other angle CDE is equal to the other angle ACB: But the angle DEC is equal to the angle AFC; therefore the remaining angle CAF is equal to the remaining angle DCE. Wherefore the triangle AFC is equiangular to the triangle DCE; and therefore (s) as CA is to AF, so is CD to CE; and by permutation, as AC is to CD, so is AF to CE. But the ratio of AC to CD is given: Therefore also the ratio of AF to CE is given. From the point A let AH be drawn perpendicular to BC: Forasmuch as the angle AFC is given, and the angle AHF also given, the third angle HAF is given: Wherefore (t) the triangle AHF is given in kind; and by consequence the ratio of AF to AH is given. But the ratio of AF to CE is also given: Therefore (u) the ratio of AH to CE is given; and therefore the ratio of the rectangle of AH and BC (x) to the rectangle of BC and CE is also given. But the ratio of the rectangle of AH and BC, to the triangle ABC is likewise given; (for the rectangle (y) is double to the triangle) and the rectangle of BC and CE is that whereby the square of the line compounded of BAC is greater than the square of BC. Therefore that space by which the square of the line compounded of BAC is greater than the square of BC has a given ratio to the triangle ADC.

Scholium.

o 4. 6.

p 5. 2.

q 5. 2.

r 17. 6.

s 4. 6.

t 40. prop.

u 8. prop.

x 1. 6.

y 41. 1.



## Scholium.

† The antient Interpreter pretending to shew the construction of the angle DEB equal to the angle ADC, saith that on the line BD and in the point D, the angle BDE ought be made equal to the angle BCD, and that the right-lines BC and DE be drawn until they intersect in E, in such sort as he supposeth the angle BCD, to be given, but it is not.

The same Interpreter afterwards shews how there may universally from a given point be drawn a right-line, given in position to a right-line, making an angle equal to a given angle: But we will also reject this way, seeing we have elsewhere shewn another more brief and easy. For example, if we would from the point D draw to the line BC given in position a right-line, making an angle equal to a given angle ADC, as is here required, we have no more to do but to assume the point K in the said line BC, and there make the triangle CKL equal to the given angle ADC: If the line KL doth meet with the point D, it shall be the line required. But if it meet not with it, from the point D let there be drawn the line DE parallel to the said KL, cutting BC prolonged in E, and the angle DEC shall be equal to the given angle ADC, for on the two parallel-lines, LK and DE, there doth fall the line BE; and therefore the angle DEC (z) is equal to the angle LKC, which hath been made equal to the given angle ADC; and by consequence the same angle DEC is also equal to ADC,

z 29. 1.

## P R O P. LXVIII. Plate VIII. Fig. 32, 33.

If two parallelograms AB and CD have to one another a given ratio, and that a side hath also a given ratio to a side, the other side shall have likewise a given ratio to the other side.

a 1. 6.

b 14. 6.

Constr. Let the ratio of BE to FD be given: I say the ratio of AE to FC is also given. For to the right-line EB let there be applied the parallelogram EH, equal to the parallelogram CD, and constituted in such sort as AE and EG may make one right-line: † Therefore KB and BH will also make one right-line.

Demonstr. Forasmuch as the ratio of AB to CD is given, and that EH is equal to the said CD; the ratio of (a) AB to EH is given; and therefore the ratio of AE to EG is also given. Seeing therefore that EH is equal

and

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and equiangled to CD, as (b) EB is to FD, so is FC to EG. But the ratio of EB to FD is given: Therefore also the ratio of FC to EG is given. But the ratio of AE to the same EG is also given: Therefore the ratio of AE to FC is given.

## Scholium.

† EUCLIDE having posited AE and EG directly in one right-line, presently concludes that KB and BH shall also make a right-line; but we shall demonstrate it thus: Seeing the lines AE and EG are posited directly, the angles AEB and BEG (c) are equal to two right-angles; and seeing that AB is a parallelogram, the lines AK and EB are parallels, on which the line AE doth fall; and therefore the two internal angles A and BEA (d) are also equal to two right-angles, and taking away the common angle BEA, there will remain the angle A, equal to the angle BEG; and consequently their opposite angles EBK and H are also equal to one another. Again, seeing that BG is a parallelogram, the two lines BE and HG are parallels, on which BH falls; and therefore the two internal angles H and EBH are equal to two right-angles. But it hath been demonstrated that H is equal to EBK: Therefore the two angles EBK and EBH are also equal to two right-angles; and therefore (e) the two lines KB and BH meet directly according to EUCLIDE.

c 13. 1.

d 29. 1.

e 14. 1.

## OTHERWISE. Plate VIII. Fig. 34, 35.

Constr. Let the given right-line K be exposed, and seeing that the ratio of A to B is given, let the same be made of K to L; therefore the ratio of K to L is also given.

f 2. prop. 1

Demonstr. But K is given; therefore (f) L is also given. Again, seeing that the ratio of CD to EF is given, let the same be made of K to M: Therefore the ratio of K to M is given. But K is given, therefore (g) M is also given; and therefore the ratio of L to M is given. Now seeing that A is equiangled to B, (b) the ratio of the said A to B is compounded of that of the sides, that is to say, CD to EF, and of CG to EH. But also the ratio of K to L is compounded of K to M, and of M to L; therefore the ratio compounded of CD to EF, and of CG to EH, is the same with that which is compounded of K to M, and of M to L (the ratio

g 1. prop.

h 23. 6.



ratio of K to L being the same as of A to B :) But the ratio of CD to EF is the same as of K to M : Therefore the other ratio of CG to EH is also the same as of M to L. But the said ratio of M to L is given : Therefore also the ratio of CG to EH is given.

P R O P. LXIX. Plate VIII. Fig. 36, 37.

*If two parallelograms, CB and EH, having the angles D and F given, and that a side hath also a given ratio to a side ; in like manner the other side shall have a given ratio to the other side.*

*Constr.* Let the ratio of BD to FH be also given : I say that the ratio of AB to EF is given. For if CB be equiangled to HE, it is manifest by the precedent Proposition ; but if it be not equiangled thereto, let the right-line DB be constituted, and in the given point B therein, let the angle DBK be made equal to the angle EFH, and finish the parallelogram DK.

*Demonstr.* Forasmuch as each of the angles BKL and BAK is given, † the other angle KBA is given : Wherefore the triangle (a) ABK is given in kind ; and therefore the ratio of AB to BK is given. But the ratio of CB to EH is supposed to be given, and (b) CB is equal to DK ; therefore the ratio of DK to EH is given ; and seeing that DK is equiangled to EH, and the ratio of the said DK to EH is given, as also that of DB to FH, (c) the ratio of BK to FE is given. But the ratio of the said BK to BA is also given : Therefore (d) the ratio of AB to FE is given.

Scholium.

† EUCLIDE supposeth here, that a parallelogram having one angle given, all the other angles are also given, and as well the antient Interpreters as others give the reasons why, the angle F being given, the other angle E shall be also given, it being the remainder of two right-angles, for that on the parallel-lines EG and FH there falls the line EF, which makes (e) the two internal angles (of the same part) F and G, equal to two right-angles. But to those angles (f) the opposite angles G and H are equal, and therefore they are also given.

From whence it follows that the angles BDC and F being given by supposition, all the other angles of the two parallelograms CB and EH, are also given : Therefore the angle DBK having been made equal to the angle F, the angle K shall be equal to the angle E, and given as that



is; But the angle BAL, which is opposite to the given angle BDC, is also given; and therefore BAK, which is the remainder of two right-angles, shall be also given; in such sort as in the triangle ABK, the two angles BAK and BKA are given, as EUCLIDE declares in this place.

P R O P. LXX. Plate VIII. Fig. 38, 39.

If of two parallelograms AB and EH, the sides about the equal angles, or about the unequal angles (yet nevertheless given angles) have to one another a given ratio, to wit, (AC to EF, and CB to FH) also the same parallelograms AB and EH shall have to one another a given ratio.

*Constr.* For let AB be prolonged to EH, and on the right-line CB let the parallelogram CM be applied equal to the parallelogram EH, in such sort as AC may be directly to CN; that is to say, that AC and CN make one right-line; and by consequence DB shall be (a) directly with BM.

a sch. 68.  
prop.

*Demonstr.* Forasmuch then as CM is equiangled and equal to EH, the sides about the equal angles shall be reciprocally (b) proportional: Wherefore as BC is to HF, so is FE to NC. But the ratio of BC to HF is given,

b 14. 6.

Therefore the ratio of FE to NC is also given. But the ratio of AC to the same EF is given: Therefore (c) the ratio of AC to NC is also given. Wherefore the ratio of AB to CM is given: (for it is, the same (d) as of AC to CN.) But CM is equal to EH: Therefore the ratio of AB to EH is given.

c 8. prop.

d 1. 6

*Constr.* Now suppose AB not to be equiangled to EH, and on the right-line CB, and in the given point C therein, let there be constituted the angle BCK, equal to the given angle F, and so finish the parallelogram CL.

*Demonstr.* Forasmuch as the angle ACB is given, and the angle BCK also given, the remaining angle ACK is given: Therefore the triangle ACK (e) is given in kind: and therefore the ratio of AC to CK is given: But the ratio of AC to EF is also given; Therefore the ratio of CK to EF is given, But the ratio of BC to HF is also given, and the angle BCK is equal to the angle F; therefore (by the first part of this proposition) the ratio of CL to EH is given. But to the said CL, AB is equal: Therefore the ratio of AB to EH is given.

e 40. prop.

U

P R O P.

## PROP. LXXI. Plate VIII. Fig. 40, 41.

If of two triangles ABC and DEF, the sides about the equal angles A and D, or else about the unequal angles (yet nevertheless given angles) have to one another a given ratio (to wit, AB to DE, and AC to DF) the same triangles shall have also to one another a given ratio ABC to DEF.

Constr. Let the parallelograms AG and DH be finished.

a 70. prop.

b 34. prop.

Demonstr. Seeing that the two parallelograms AG and DH, have the sides about the equal angles A and D, or else about the unequal angles (nevertheless given) in a given ratio to one another, the ratio (a) of the parallelogram AG to the parallelogram DH is given. But the triangle ABC is the half of the parallelogram AG (b) and the triangle DEF the half of the parallelogram DH. Therefore the ratio of the triangle ABC to the triangle DEF is given.

## PROP. LXXII. Fig. 42, 43.

If of two triangles ABC and DEF, the bases BC and EF, are in a given ratio, BC to EF, and that from the angles A and D, there be drawn to those bases the right-lines AG and DH, making the angles AGC and DHF equal, or unequal (yet nevertheless given) which shall have to one another given ratio's AG to DH, those triangles ABC and DEF shall have also a given ratio to one another, to wit, ABC to DEF.

Constr. For let the parallelograms KC and LE be finished.

a 29. 1.

b 70 prop.

Demonstr. Forasmuch as the angles AGC and DHF are equal, or unequal (yet given) and that the angle AGC (a) is equal to the angle KBC, and also the angle DHF equal to the angle LEF, the angles at the points C and E are equal, or else unequal (yet given,) and because the ratio of AG to DH is given, and AG is equal to KC, and DH is equal to LE, therefore the ratio of KC to LE is given. But the ratio of BC to EF is also given (b) and the angles at the points B and E are equal, or unequal (yet given :) Therefore (b) the ratio of the triangles ABC to DEF is given.

parallelogram KC to the parallelogram LF is given ; and therefore the ratio of the triangle ABC to the triangle DEF is given, seeing those triangles (c) are the one half of the parallelograms.

c 41. 1.

P R O P. LXXIII. Plate VIII. Fig. 38, 39.

*If of two parallelograms AB and EH, the sides about the equal angles C and F, or else about the unequal angles (but nevertheless given) are in such sort to one another, that as the side CB of the first, is to the side EH of the second ; so the other side EF of the second, is to some other right-line CN. But that the other side AC, hath also to the same right-line CN a given ratio, those parallelograms will have also to one another a given ratio AB to EH.*

*Constr.* For in the first place, let the parallelogram AB be equiangled to EH, and having placed CN directly to AC : Let the parallelogram CM be finished.

*Demonst.* Forasmuch then as CB or NM its equal, is to FH, so is EF to CN, and that the angles N and F are equal (for N is equal to the angle ACB, which is put equal to F) the parallelograms CM and EH (a) are equal : But as AC to CN, so (b) the parallelogram AB is to the parallelogram CM or EH : Therefore seeing that the ratio of AC to CN is given, the ratio of AB to EH is also given.

a 14. 6.

b 1. 6.

*Constr.* 2. Now suppose the parallelogram AB not to be equiangled to the parallelogram EH, and let there be constituted at the given point C in the line CB, the angle BCK, equal to the angle EFH, and so finish the parallelogram CL.

*Demonst.* 2. Seeing that each of the angles ACB and KCB is given, the remaining angle ACK is also given. But (c) the angle CAK is given, as also the remaining angle AKC : Therefore (d) the triangle ACK is given in kind : and therefore the ratio of AC to CK is given. But the ratio of the same AC to CN is also given : Therefore (e) the ratio of CK to CN is given. And seeing that as CB is to FG ; so is EF to the right-line CN, to which the other side KC hath a given ratio, and that the angle BCK is equal to the angle F, the ratio of the parallelogram CL to the parallelogram EH is given (by the first part of this proposition) but the parallelogram CL is equal to the parallelogram AB : Therefore the ratio of the parallelogram AB to the parallelogram EH is given.

c sch. 69.

prop.

d 40. prop.

e 8. prop.

U 2

P R O P.



## P R O P. LXXIV. Plate VIII. Fig. 38, 39.

If two parallelograms AB and EH, in equal angles C and F, or else in unequal angles (yet nevertheless given angles) have a given ratio to one another, as one side CB of the first shall be to one side FH of the second, so the other side EF of the second, shall be to that to the which the other side AC of the first hath a given ratio.

*Constr.* For either AB is equiangled or not; suppose it in the first place to be equiangled, and to the right-line BC let there be applied the parallelogram CM, equal to the parallelogram EH, and so posited, as that AC and CN may be direct: Therefore (a) DB and BM shall be also direct (that is, as one right-line.)

*Demonstr.* Seeing that the ratio of AB to EH is given, and that CM is equal to EH, the ratio of AB to CM is also given; and therefore the ratio of AC to CN is given (seeing AB is to CM, (b) as AC is to CN;) and as CM is equal and equiangled to EH, the sides about the equal angles of the parallelograms CM and EH. (c) are reciprocally proportional; and therefore as CB is to FH, so is EF to CN. But the ratio of AC to CN is given: Therefore as CB is to FH, so is EF to that to which AC hath a given ratio.

*Constr.* 2. Now suppose AB not to be equiangled to EH, and in the given point C of the line CB, let there be constituted the angle BCK equal to the angle EFH, and finish the parallelogram CL.

*Demonstr.* 2. Seeing then that the ratio of AB to EH is given, and (d) that AB is equal to CL, also the ratio of CL to EH is given, and the angle BCK is equal to the angle F, and therefore CL (e) is equiangled to EH: Therefore (by the first part of this proposition) as CB is to FH, so is EF to that to the which CK hath a given ratio. But the ratio of AC to CK is given; (as appears by what hath been demonstrated in the latter part of the precedent proposition.) Therefore as CB is to FH, so is EF to that to which AC hath a given ratio,

a sch. 68.  
prop.

b 1. 6.

c 14. 6.

d 36. 1.

e sch. 69.  
prop.

P R O P.

PROP. LXXV. Plate VIII. Fig. 40, 41.

*If two triangles ABC and DEF, in equal angles A and D, or else unequal (yet nevertheless given) have to one another a given ratio, as the side AB of the first, shall be to the side DE of the second, so the other side DF of the second, shall be to that right-line to the which the other side AC of the first hath a given ratio.*

*Constr.* For let the parallelograms AG and DH be finished.

*Demonstr.* Forasmuch as the ratio of the triangle ABC to the triangle DEF is given, also the ratio of the parallelogram AG to the parallelogram DH is given.

Seeing therefore that the two parallelograms AG and DH in equal angles, or unequal angles (nevertheless given) have to one another a given ratio; as (a) AB is to DE, so is DF to that to which AC hath a given ratio.

a 74. prop.

PROP. LXXVI. Fig. 4.

*If from the top A of a triangle ABC, given in kind, there be drawn to the base BC, a perpendicular line AD, that line AD shall have to the base BC a given ratio.*

*Demonstr.* For seeing that the triangle ABC is given in kind, the ratio of AB to BC is given; and the angle B is also given. But the angle ADB is given; therefore the other angle BAD is given. Wherefore (a) the triangle ADB is given in kind: and therefore the ratio of AB to AD is given. But the ratio of AB to BC is given: Therefore (b) the ratio of AD to BC is given.

a 40. prop.

b 8. prop.

PROP. LXXVII. Fig. 44, 45.

*If two figures ABC and DEF, given in kind, have to one another a given ratio, the ratio also shall be given of which you please of the sides of one of the figures, to which you please of the sides of the other figure.*

*Constr.* For on the right-lines BC and EF, let there be described the squares BG and EH.

a 49. prop.

b 8. prop.

*Demonstr.* Forasmuch as on one and the same right-line BC, are described two figures ABC and BG given in kind, (a) the ratio of the said ABC to BG is given. In like manner the ratio of DEF to EH is given; and seeing that the ratio of ABC to DEF is given; and also that of the same figure ABC to BG; and again the ratio of DEF to EH: (b) the ratio of BG to EH is given; and therefore the ratio of BC to EF is also given.

P R O P. LXXVIII. Plate VIII. Fig. 46, 47.

*If a given figure ABC, hath a given ratio to some rectangled figure DF, and that one side BC hath a given ratio to one side DE, the rectangled figure DF is given in kind.*

a sch. 68. prop.

b 49. prop.

c 8. prop.

d 14. 6.

e 1. 6.

f 8. prop.

g sch. 61. prop

*Constr.* For on the right-line BC let the square BH be described, and to the right-line DE, let the parallelogram DK be applied equal to BH, in such a manner, as that GD and DI may be placed directly, (a) and by consequence FE and EK also directly.

*Demonstr.* Therefore seeing that on one and the same right-line BC are described the two rectiline figures ABC and BH, given in kind, (b) the ratio of ABC to BH is given. But the ratio of the said ABC to DF is also given: Therefore (c) the ratio of BH to DF is given. But BH is equal to DK: Therefore the ratio of DK to DF is also given. And seeing that BH is equal and equiangular to DK, both the one and the other being rectangles, (d) the sides of those figures are reciprocally proportional; and as BC is to DE, so is DI to CH. But by supposition, the ratio of BC to DE is given; therefore also the ratio of DI to CH is given; but the ratio of DI to DG is also given: (for DI is to DG (e) as DK to DF:) Therefore (f) the ratio of DG to CH is given. But CH is equal to BC, seeing that BH is a square; therefore the ratio of the same BC to DG is given: But the ratio of the same BC to DE is also given; therefore the ratio of DE to DG is given, and the angle at D is a right angle: Therefore (g) DF is given in kind.

P R O P.



P R O P. LXXIX. Plate VIII. Fig. 48, 49.

If two triangles  $ABC$  and  $EFG$ , have an angle  $B$  equal to an angle  $F$ . And from the equal angles  $B$  and  $F$  there be drawn perpendiculars  $BD$  and  $FH$ , to the bases  $AC$  and  $EG$ ; and that as the base  $AC$  of the first triangle  $ABC$ , is to the perpendicular  $BD$ , so also the base  $EG$  of the other triangle  $EFG$ , is to the perpendicular  $FH$ , those triangles  $ABC$  and  $EFG$  are equiangled.

Constr. For about the triangle  $EFG$  let there be described the circle  $EFLG$ , then on the right-line  $EG$ , and in the point  $E$  given therein, let there be made the angle  $GEL$ , equal to the angle  $C$ , and let  $FL$  and  $LG$  be drawn, and the perpendicular  $LM$ .

Demonstr. Seeing then that the angle  $GEL$  is equal to the angle  $C$ , and the angle  $ELG$  is equal to the angle  $EFG$ , (a) they being in one and the same segment of the circle; the third angle  $EGL$  is equal to the third angle  $A$ . Wherefore the triangle  $ABC$  is alike to the triangle  $ELG$ , and the perpendiculars  $BD$  and  $LM$  are drawn: Therefore † as  $AC$  is to  $BD$ , so is  $EG$  to  $LM$ ; but by supposition as  $AC$  is to  $BD$ ; so is  $EG$  to  $FH$ : Therefore (b)  $LM$  is equal to  $FH$ . But the said  $LM$  is (c) parallel to  $FH$ : Therefore (d)  $FL$  is also parallel to  $EG$ ; and therefore the angle  $FLE$  (e) is equal to the angle  $LEG$ . But the angle  $C$  is also equal to the said angle  $LEG$ , and the angle  $FLE$  to the angle  $FGE$  (f): Therefore also the angle  $C$  is equal to the angle  $FGE$ . But by supposition the angle  $ABC$  is equal to the angle  $EFG$ : Therefore the third angle  $BAC$  is equal to the third angle  $FEG$ : Wherefore the triangle  $ABC$  is equiangled to the triangle  $EFG$ .

Scholium.

† Now that as  $AC$  is to  $BD$ , so  $EG$  is to  $LM$ , it is by some thus demonstrated. Forasmuch as the angle  $C$  is equal to the angle  $GEL$ , and the angle  $BDC$  to the angle  $LME$ , each being a right-angle, the other angle  $CBD$  is equal to the other angle  $ELM$ : Therefore (g) as  $EM$  is to  $ML$ , so is  $CD$  to  $DB$ . Again, seeing the angle  $ABC$  is equal to the angle  $ELG$ , and the angle  $CBD$  to the angle  $ELM$ , the remaining angle  $ABD$  is equal to the remaining angle  $MLG$ ; but the angle  $ADB$  is also equal to the angle  $LMG$ ; and therefore the third angle  $A$  is equal to the

U 4

third

a 21. 3.

b 7. 5.

c 28. 1.

d 33. 1.

e 29. 1.

f 21. 3.

g 4. 6.

h 4. 6.

i 14. 5.

third angle LGM : Therefore (h) as AD is to DB, so is GM to ML. But it hath been demonstrated, that as CD is to DB, so is EM to ML : Therefore (i) as AC is to BD, so is EG to LM.

## P R O P. LXXX. Plate VIII. Fig. 50, 51.

If a triangle ABC hath one angle A given, and that the rectangle contained under the sides AB and AC, comprising the given angle A, hath a given ratio to the square of the other side BC, the triangle ABC is given in kind.

Constr For from the points A and B, let there be drawn the perpendiculars AD and BE.

a 40. prop.

b 1. 6.

c 41. 1.

d 8. prop.

e 1. 6.

f 8. def.

g 4. def.

h 2. prop.

i 27. prop.

k 28. prop.

l 25. prop.

Demonstr. Forasmuch as the angle BAE is given, and also the angle AEB, the triangle ABE is given in (a) kind ; and therefore the ratio of AB to BE is given : Therefore the ratio of the rectangle of AB and AC to the rectangle of BE and AC is also given (for it is the same ratio (b) as of AB to BE.) But the rectangle of AC and BE is equal to the rectangle of BC and AD ; for that each of those rectangles is (c) double to the triangle ABC. Therefore the ratio of the rectangle of AB and AC to the rectangle of BC and AD is also given. But the ratio of the rectangle of AB and AC to the square of BC is given : Therefore (d) also the ratio of the rectangle of BC and AD to the square of BC is given ; and therefore the ratio of the right-line BC to the right-line AD is given. (For that (e) the rectangle is to the square as AD to BC.) Now let the right-line FD, given in position and magnitude, be exposed ; and thereon let there be described the segment of a circle FID ; capable of an angle equal to the angle A. And seeing the said angle A is given, also the angle in the segment FLD shall be given ; and therefore (f) the same segment is given in position. From the point D let there be erected at right-angles on the line FD, the line DH, which (g) is given in position : Let it be so made, that as BC is to AD, so FD may be to DH and seeing that the ratio of BC to AD is given, also that of FD to DH is given. But FD is given : Therefore (h) DH is given in magnitude. But it is also given in position, and the point D is given : Therefore the point H is (i) also given. Now through the point H let there be drawn HI, parallel to FD, and that line HI shall be given in (k) position. But the segment of the circle FID is also given in position. Therefore (l) the point

point I is given. Let the right-lines IF and ID be drawn, and the perpendicular IE: Therefore IE is given in position. But the point I is given, as also each of the points F and D: Therefore (m) each of the lines FD, FI, and ID is given in position and magnitude: Wherefore (n) the triangle FID is given in kind; and seeing that as BC is to AE, so is FD to DH, and (o) that to DH, IE is equal; as BC is to AE, so is FD to IE, and the angle A is equal to the angle FID: Therefore (p) the triangle ABC is equiangled to the triangle FID. But FID is given in kind: Therefore also the triangle ABC is given in kind.

m 26. prop.

n 39. prop.

o 34. prop.

p 79. prop.

OTHERWISE. Fig. 52, 53.

*Constr.* Let the triangle ABC, whose angle A is given, and the ratio of the rectangle contained under AB and AC, to the square of BC be given: I say that the triangle ABC is given in kind.

*Demonstr.* For seeing the angle A is given, that space by which the square of the line compounded of BAC is greater than the square of BC, (q) hath a given ratio to the triangle ABC. Now let that space be D: Therefore the ratio of D to the triangle ABC is given. But the ratio of the triangle ABC to the rectangle of AB and AC is given; (r) seeing the angle A is given: Therefore (s) the ratio of the space D to the rectangle of AB and AC is given. But the ratio of the rectangle of AB and AC to the square of BC is also given: Therefore (s) the ratio of the space D to the square of BC is given. Wherefore by compounding, (t) the ratio of the space D, with the square of BC the said square of BC is given: Therefore the ratio of the square of the line compounded of BAC, to the square of BC is given; (for that the space D with the square of BC is equal to the square of the line compounded of BAC;) and therefore (u) the ratio of the said line compounded of BAC to BC is given. But the angle A is also given: Therefore (x) the triangle ABC is given in kind.

q 67. prop.

r 66. prop.

s 8. prop.

t 6. prop.

u sch. 52.

prop.

x 46. prop.

PROP. LXXXI.

If of three right-lines A, B, and C, proportional to three other proportional right-lines D, E, and F, the extremes A and D, C and F, are in a given ratio, (to wit, as A to D, and C to F,) also the means, B and E shall be in a given ratio, and if one extreme hath a

| A | D     |
|---|-------|
| B | E     |
| C | F     |
|   | given |



given ratio to an extreme, and the mean to the mean, the other will have also a given ratio to the other.

*Demonstr.* Forasmuch as the ratio of A to D, and of C to F is given, the rectangle of A and D (a) shall have a given ratio to the rectangle of C and F. But the rectangle of A and D is equal (b) to the square of B; and the rectangle of C and F to the square of E. Therefore the ratio of the square of B to the square of E is given; and therefore (c) the ratio of the line B to the line E is also given.

Again, let the ratio of A to D, and B to E, be given: I say that the ratio of C to F is also given. For seeing that the ratio of A to D, and of B to E is given, also the ratio of the square of B (d) to the square of E is given. But the square of B is equal to the rectangle of A and C, and the square of E to the rectangle of D and F. Therefore the ratio of the rectangle of A and C to the rectangle of D and F is given. But the ratio of a side A to a side D is given: Therefore (e) the ratio of the other side C to the other side F is also given.

## PROP. LXXXII.

If there be four right-lines A, B, C, and D, proportional, as the first A shall be to that line to which the second B hath a given ratio, so the third C shall be to that to which the fourth D hath a given ratio.

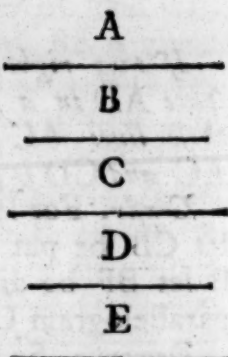
*Constr.* Let E be the line to which B hath a given ratio, and let it be so as that B may be to E, as D is to F.

*Demonstr.* Now the ratio of B to E is given, therefore also the ratio of D to F is given. And seeing that as A is to B, so is C to D. And again, as B is to E, so is D to F, by ratio of equality, as A is to E, so is C to F. But E is that line to which B hath a given ratio, and F that to which D also hath a given ratio: Therefore as A is to that to which B hath a given ratio, so C is to that to which D hath a given ratio.

PROP.

PROP. LXXXIII.

If four right-lines A, B, C, and D, are in such sort to one another, that of any three of them A, B, C, and a fourth E, taken proportional, to which that line D, which remains of the four lines, hath a given ratio, it shall be as the fourth D is to the third C, so the second B shall be to that to which the first A hath a given ratio.



*Demonstr.* Forasmuch as A is to B, as C is to E, the rectangle contained under A and E (a) is equal to the rectangle contained under B and C; and seeing that the ratio of D to E is given, also shall be given the ratio of the rectangle of A and D to the rectangle of A and E (for (b) it is the same ratio as of D to E.) But the rectangle of A and E is equal to the rectangle of B and C. Therefore the ratio of the rectangle of A and D to the rectangle of B and C is given. Wherefore (c) as D is to C, so is B to that to which A hath a given ratio.

a 16. 6.

b 1. 6.

c 56. prop.

PROP. LXXXIV. Plate VIII. Fig. 54.

If two right-lines AB and AE comprehending a given space AF in a given angle BAE, and that the one AB be greater than the other AE by a given line CB, also each of the lines AB and AE is given.

*Demonstr.* For seeing that AB is greater than AE by the given line CB, the remainder AC is equal to AE: Finish the parallelogram AD. Therefore seeing that AE is equal to AC, the ratio of AE to AC is given, and the angle A is also given: Therefore (a) AD is given in kind. Wherefore the given space AF is applied to the given right-line CB, exceeding it by the given figure AD given in kind; and therefore (b) the breadth of the excess is given. Therefore AC is given. But CB is also given: Therefore the whole AB is given. But AE is also given: Therefore each of the right-lines AB and AE is given.

a sch. 61. prop.

b 59. prop.

PROP.

## PROP. LXXXV. Plate VIII. Fig. 54.

If two right-lines AC and CD, comprehend a given space AD in a given angle ACD, the line compounded of those lines AC and CD is given, also each of those lines AC and CD is given.

*Constr.* For let AC be prolonged to the point B, and let CB be put equal to CD, then through the point B let BF be drawn parallel to CD, and so finish the parallelogram CF.

*Demonstr.* Seeing then that CB is equal to CD, and the angle DCB is given; for that angle that follows is the given angle; and therefore (a) the parallelogram DB is given in kind: and again, seeing that the line compounded of ACD is given, and CB is equal to CD, also AB is given. And thus to the right-line AB there is applied the given space AD, deficient by the figure DB given in kind: and therefore (b) the breadths of the defects are also given: Therefore the right-lines DC and CB are given. But the compounded line ACD is also given: Therefore (c) each of the lines AC and CD is given.

a sch. 61.  
prop.

b 58. prop.

c 4. prop.

## PROP. LXXXVI. Fig. 55.

If two right-lines AB and BC, comprehend a given space AC, in a given angle ABC, the square of the one BC, is greater than the square of the other AB, by a given space (yet in a given ratio,) also each of those lines AB and BC shall be given.

*Demonstr.* For seeing that the square of BC is greater than the square of AB by a given space (yet in a certain ratio:) Let the given space be taken away, that is to say, the rectangle contained under CB and BE: Therefore (a) the ratio of the remainder, (b) which is the rectangle contained under BC and CE to the square of AB is given. And forasmuch as the rectangle † under AB and BC is given, and also that of CB and BE, their (c) ratio is given. But as the rectangle under AB and BG is to the rectangle under CB and EB, (d) so AB is to BE; and therefore the ratio of AB to BE is given: Wherefore (e) the ratio of the square of AB to the square of BE is also given. But the ratio of the square of AB to the rectangle under BC and CE is given: Therefore (f) also the ratio of the rectangle under BC

a 11. def.  
b 2. 2.

c 1. prop.  
d 1. 6.

e 50. prop.

f 8. prop.

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BC and CE to the square of BE is given. . Wherefore the ratio of four times the rectangle under BC and CE to the square of BE is given; and by compounding, (g) the ratio of four times the rectangle under BC and CE, with the square of BE, the square of BE is given. But four times the rectangle of BC and CE, with the square of BE, (h) is the square of the compound line BCE: Therefore the ratio of the square of the compound line BCE to the square of BE is given: Wherefore (i) the ratio of the line compounded of BC and CE to BE is given, and by compounding (k) the ratio of the compound of the lines BC, CE, and BE, that is to say, the double of BC to BE is given; and therefore the ratio of the only line BC to BE is also given. But as BC is to BE, (l) so the rectangle under BC and BE is to the square of BE: Therefore the ratio of the rectangle under BC and BE to the square of BE is given. But the rectangle of BC and BE is given: Therefore (m) the square of BE is also given, and consequently the line BE is given. Wherefore BC is also given, seeing that the ratio of BE to BC is given. But the space AC is given, and also the angle B: Therefore (n) AB is given. Wherefore each of the lines AB and BC is given.

g 6. prop.

h 8. 2.

i 54. prop.

k 6. prop.

l 1. 6,

m 2. prop.

n 57. prop.

Scholium.

† Instead of saying in this place [what is under, &c.] we have used this word rectangle, it being manifest by what follows that such was the intention of EUCLIDE, seeing that he makes use in the said Demonstration of the second, and eighth proposition of the second Element; and also that the space or parallelogram being given not rectangled, it may be reduced thereto, making on BC, and in the given point B, a right-angle CBA, so as that there will be two parallelograms constituted on one and the same base BC, and between the same parallels, as in the 69th proposition, by means whereof this conclusion is drawn.

Note, This serves also for the next Prop.

P R O P. LXXXVII. Fig. 56.

If two right-lines AB and BC, comprehend a given space AC, in a given angle B, the square of the one BC is greater than the square of the other AB, by the given space; also each of those lines AB and BC shall be given.

Demonstr.

- Demonstr.* For seeing that the square of BC is greater than the square of AB by a given space: Let the given space be taken away, and let the rectangle be contained under BC and BE: Therefore the remainder (a) which is the rectangle of BC and CE, is equal to the square of AB. And seeing that the rectangle of BC and BE is given, and also the space or rectangle AC, the ratio of the said rectangle of BC and BE to AC is given. But as (b) the rectangle of BC and BE is to the rectangle of AB and BC, so is BE to AB: Therefore the ratio of BE to AB is given, and therefore (c) the ratio of the square of the said DE to the square of AB is also given. But to that square of AB the rectangle of BC and CE is equal: Therefore the ratio of the said rectangle of BC and CE to the square of BE is given; and therefore the ratio of the quadruple of the said rectangle of BC and CE to the square of BE is also given; and by compounding (d) the ratio of four times the rectangle of BC and CE, with the square of BE to the said square of BE is given. But four times the rectangle of BC and CE, with the square of BE, (e) is the square of the compound line BCE: Therefore the ratio of the square of that compound line BCE to the square of BE is also given; and therefore the ratio (f) of the compound line BCE to BE is given. Wherefore by compounding (g) the ratio of the said compound line BCE and EB, that is to say, twice BC to BE is also given; therefore the ratio of the only line BC to BE is given. But the ratio of the same BE to AB is also given: Therefore (h) the ratio of AB to BC is given. And seeing that the ratio of BC to BE is given, and that as the said BC is to BE, so is the square of BC (i) to the rectangle of BC and BE, the ratio of the square of BC to the rectangle of BC and BE is also given. But the said rectangle of BC and BE is given, it being that which was taken away, and which was given. Therefore the square of BC (k) is given, and therefore the line BC is given. But the ratio of the same BC to BA is given, therefore AB is also given.
- a 2. 2.  
b 1. 6.  
c 30. prop.  
d 6. prop.  
e 8. 2.  
f 54. prop.  
g 6. prop.  
h 8. prop.  
i 1. 6.  
k 2. prop.

P R O P. LXXXVIII. Plate VIII. Fig. 57.

*If in a circle ABC, given in magnitude, there be drawn a right-line AC, which shall take away a segment ABC, which doth comprehend a given angle AEC, that line AC is given in magnitude.*

I

*Constr.*



*Constr.* For let D be the center of the circle ; and let the diameter thereof, ADE be drawn, and let EC be joined.

*Demonstr.* The angle ACE is given for (a) it is a right-angle. But the angle AEC is also given, and therefore the other angle CAE is given. Wherefore the triangle ACE (b) is given in kind ; and therefore the ratio of EA to AC is given. But AE is given in magnitude, seeing that the circle ABC is given in magnitude. Therefore (c) AC is also given in magnitude.

a 31. 3.

b 40. prop.

c 2. prop.

P R O P. LXXXIX. Plate VIII. Fig 57.

*If in a circle ABC, given in magnitude, there be drawn a right-line AC, given in magnitude, that line AC will take away a segment ABC, comprehending a given angle.*

*Constr.* For having taken the point D for the center of the circle, let the diameter ADE be drawn, as also the right-line EC.

*Demonstr.* Forasmuch as each of the right-lines AE and AC are given, the ratio of the line AE to AC (a) is given ; and the angle ACE is a right-angle : Therefore (b) the triangle ACE is given in kind, and therefore the angle AEC is given.

a 1. prop.

b 43. prop.

P R O P. XC. Fig. 58.

*If in the circumference of a circle ABC, given in position and in magnitude, there be taken a given point B, and that from the point B to the circumference of the circle ABC, a right-line BAC be inflected so as to make a given angle BAC, the other extremity C of the inflected-line shall be given.*

*Constr.* For let the center of the circle be D, and let the right-lines BD and BC be drawn.

*Demonstr.* Forasmuch as each point B and D is given, the right-line BD, (a) is given in position ; and seeing that the angle BAC is given, the angle BDC is also given. Wherefore to the right-line BD, given in position, and in the point D given therein, there is drawn the right-line CD ; which makes the given angle BDC ; and therefore (b) the line DC is given in position. But the circle ABC is given in position and magnitude : Therefore (c) the right-line DC is given in position and in magnitude. But the point is D given : Therefore (d) the point C is also given.

a 26. prop.

b 29. prop.

c 6. def.

d 27. prop.

P R O P.



## P R O P. XCI. Plate VIII. Fig. 59.

*If from a given point C, there be drawn a right-line CA, which shall touch a circle AB, given in position; that line CA is given in position and in magnitude.*

*Constr.* For having taken the point D for the center of the circle, let the right-lines DA and DC be drawn.

*Demonstr.* Forasmuch as each point C and D is given, the right-line CD (*a*) is given in position and in magnitude. But the angle CAD (*b*) is a right-angle; and therefore the semicircle described on CD shall pass through the point A: Let it then pass through that point, and let the semicircle be DAC: Forasmuch as the same DAC (*c*) is given in position, and also the circle ABE, (*d*) the point A is given. But the point C is also given: Therefore (*e*) the right-line AC is given in position and in magnitude.

a 26. prop.  
b 18. 3.

c 6 def.  
d 25. prop.  
e 26. prop.

## P R O P. XCII. Fig. 60.

*If without a circle ABC, given in position, there be taken some point D, and from that given point there be drawn a right-line DB, cutting the circle, the rectangle comprised under the whole line BD, and the part DC, between the point D, and the convexity of the circumference AC, shall be given.*

*Constr.* For from the point D let the right-line DA be drawn, which shall touch the circle in the point A.

*Demonstr.* Therefore DA (*a*) is given in position and magnitude; and therefore the square of the said DA is (*b*) given. But the said square of DA is equal (*c*) to the rectangle of BD and DC: Therefore the said rectangle of BD and DC is also given.

a 91. prop.

b 52. prop.  
c 36. 3.

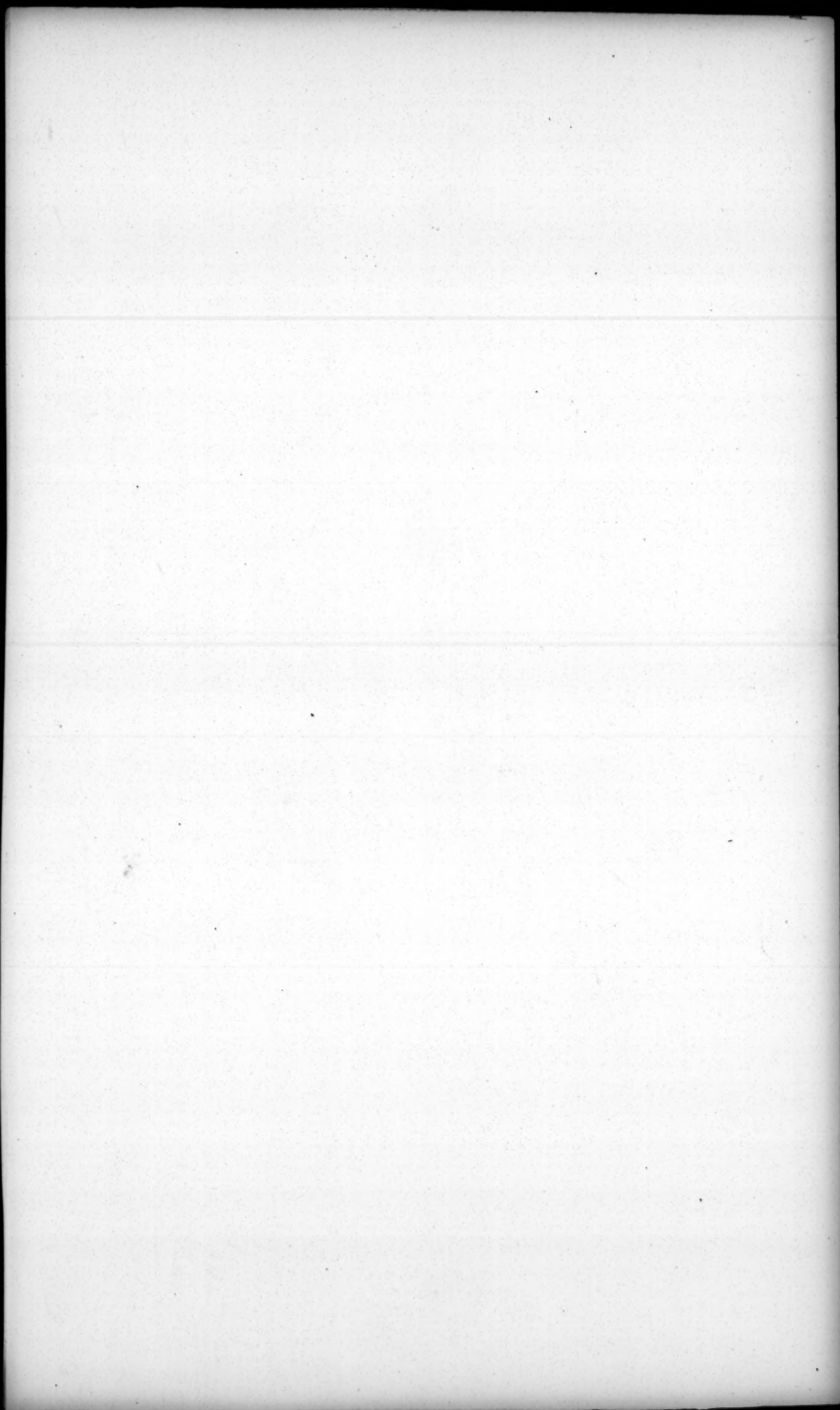
## O T H E R W I S E. Fig. 61.

*Constr.* Let E be the center of the circle, and through the same center let there be drawn from the point D the right-line DA.

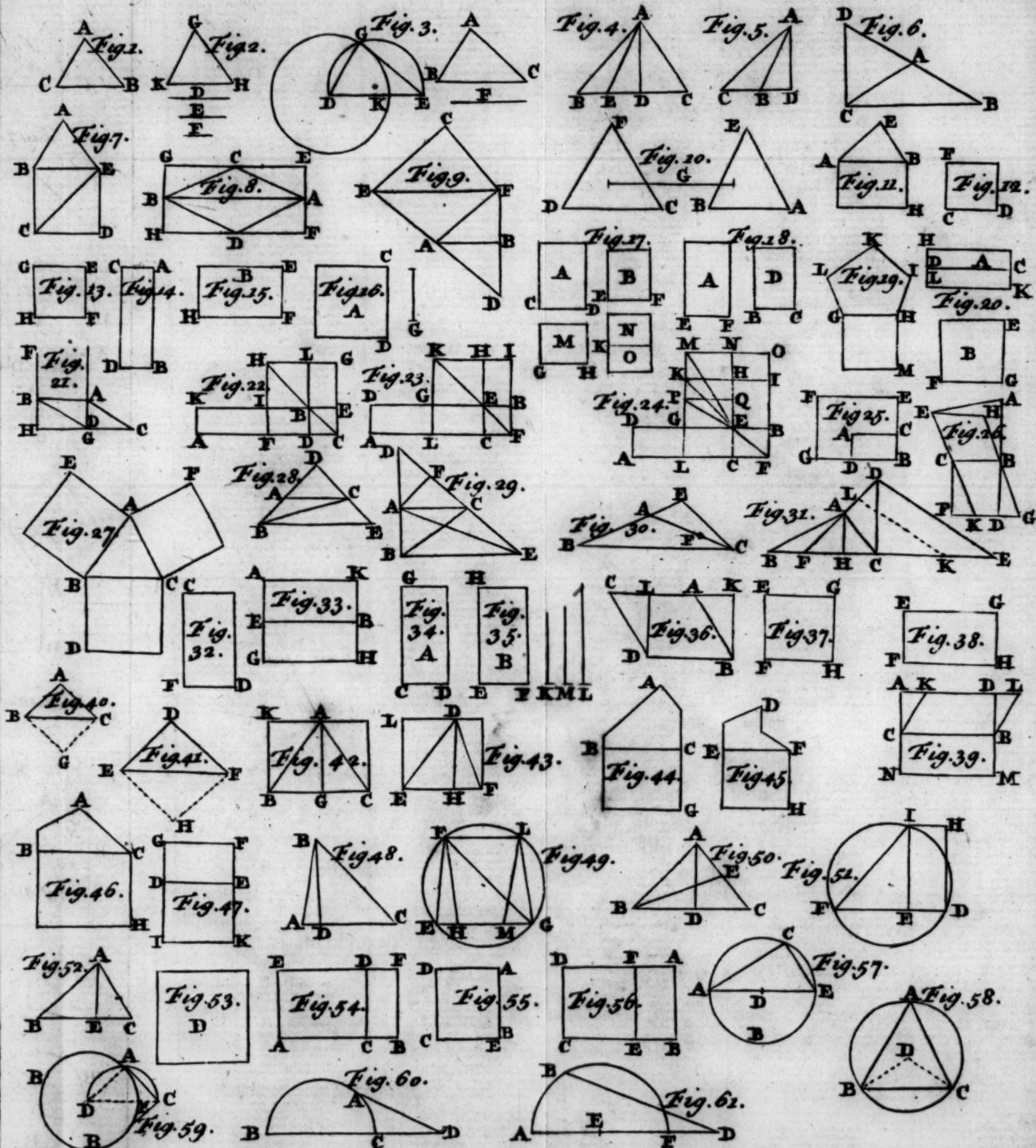
*Demonstr.* Forasmuch as each point D and E is given, the right-line DE is (*d*) given in position and in magnitude. But the circle ABC is given in position and in magnitude: Therefore each point A and F (*e*) is given, and the point D is also given; and therefore each line AD

d 26. prop.

e 25. prop.











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AD and FD is given. Wherefore the rectangle of the lines AD and DF is also given. But the said rectangle of AD and DF is equal to the rectangle of DB and DC: Therefore the rectangle of DB and DC is given.

P R O P. XCIII. Plate IX. Fig. 1.

*If in a circle given in position there be taken a given point A, and through that point A there be drawn a right-line BC to [the circle, the rectangle comprised under the segments the same line BC, shall be given.*

*Constr.* For let D be taken for the center of the circle, and having drawn the right-line AD prolong it to the point E and F.

*Demonstr.* Forasmuch as each point A and D is given, the right-line AD (a) is given in position. But the circle BEC is also given in position: Therefore each point E and F is also given in position, and the point A is given. Wherefore each line (b) AE and AF is given: Therefore the rectangle of the same lines AE and AF is given, and is equal to the rectangle (b) of AB and AC: Therefore the said rectangle of AB and AC is given.

a 26. prop.

b 35. 3.

P R O P. XCIV. Fig. 2.

*If in a circle ABC, given in magnitude, there be drawn a right-line BC, which takes away a segment which comprehends a given angle BAC, and that the said angle in the segment is cut into two equal parts, the line compounded of the right-lines BA and AC, which comprehend the given angle BAC shall have a given ratio to the line AD, which divides that angle into two equal parts; and the rectangle contained under the line compounded of those lines BA and AC, comprehending the given angle BAC, and that part ED of the intersecting-line which is below the segment between the base BC and the circumference, shall be given.*

*Constr.* Let BD be drawn.

*Demonstr.* Forasmuch as in the circle ABC given in magnitude, there is drawn the right-line BC, which takes away the segment BAC, comprehending the given angle BAC, that line BC (a) is given; and therefore BD is also given: Therefore the ratio of BC to BD (b) is given. And seeing that the given angle BAC is cut in two equal parts by the right-line AD, as (c) BA is to CA, so is BE to CE; and by compounding, as

a 88. prop.

b 1. prop.

c 3. 6.

X

BAC

d 21. 3. BAC is to CA, so is BC to CE; and by permutation, as BAC is to BC, so is CA to CE. And seeing that the angle BAE is equal to the angle CAE, and the angle ACE (*d*) to the angle BDE, the other angle AEC is equal to the other angle ABD; and therefore the triangle ACE is equiangled to the triangle ABD: Therefore (*e*) as AC is to CE, so is AD to BD. But as CA is to CE, so the line compounded of BA and AC is to BC. Therefore as the compound-line BAC is to BC, so is AD to BD; and by permutation, as the compound-line BAC is to AD, so is BC to BD. But the ratio of BC to BD is given: Therefore the ratio of the compound-line BAC to AD is also given. Moreover, I say that the rectangle under the compound-line BAC and ED is given. For seeing that the triangle AEC is equiangled to the triangle BDE, (for the angle ACE (*d*) is equal to the angle BDE, and the angle AEC (*f*) to the angle BED) as BD is to DE, so is AC to CE. But as AC is to CE, so is also the compound-line BAC to BC: Therefore as the compound-line BAC is to BC, so is BD to DE. Wherefore the rectangle of the compound-line BAC and DE (*g*) is equal to the rectangle of BC and BD. But the rectangle of BC and BD is given, (for that those lines BC and BD are given:) Therefore the rectangle under the compound-line BAC and ED is also given.

f 15. 1.

g 16. 6.

## O T H E R W I S E. Plate IX. Fig. 4.

*Constr.* Let CA be prolonged to the point E, and let AE be put equal to BA, and let BE and BD be joined.

h 32. 1.

i 5. 1,

k 21. 3.

*Demonstr.* Forasmuch as the angle BAC is double to each of the angles CAD and AEB (for the angle BAC is cut into two equal parts by the line AD, and equal (*b*) to the two angles ABE and AEB, which (*i*) are equal) the angle ABE is equal to the angle CAD, that is to say, (*k*) to the angle CBD; adding therefore the common angle ABC, the whole angle ABD shall be equal to the whole angle FBE. But the angle ACB is (*k*) equal to the angle ABD: Therefore the third angle AEB is equal to the third angle BAD; and therefore the triangle CEB is equiangled to the triangle ABD: Wherefore as CE is to CB, so is AD to BD. But the right-line CE is compounded of the two lines CA and AB: Therefore as the compound-line BAC is to CB, so is AD to BD; and by permutation,

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as the compound-line BAC is to AD, so is CB to BD. But the ratio of CB to BD is given, seeing that each of those lines is given: Therefore the ratio of the compound-line BAC to AD is also given. And seeing that the triangle CEB is equiangular to the triangle FBD (for the angle AFC is equal (*l*) to the angle BFD, the angle ECB (*m*) to the angle ADB) as EC is to CB, so is BD to DF. But EC is equal to the compound-line BAC: Therefore as the compound-line BAC is to CB, so is BD to DF. Wherefore (*n*) the rectangle of the compound-line BAC and DF is equal to the rectangle of CB and BD. But the rectangle of CB and DC is given, since each of the lines CB and BD is given; Therefore the rectangle of the compound-line BAC and DF is given.

l 21. 3.  
m 16. 6.

n 16. 6.

OTHERWISE. Plate IX. Fig. 5.

*Constr.* Let AC be prolonged to F, and let CF be put equal to AB, and let the right-lines BD and DF be drawn.

*Demonstr.* Forasmuch as BA is equal to CF, and (*o*) BD to DC, the two sides AB and BD are equal to the two sides CD and DF, each to his corresponding side, and the angle ABD is equal to the angle DCF, (*p*) seeing that the four sided figure ABDC is within the circle: Therefore the base AD is (*q*) equal to the base DF, and the angle DAB to the angle DFC. But the angle BAD is given, being the half of the given angle BAC, therefore the angle DFC is so also. But DAF is also given: Therefore the triangle ADF is given in kind. Wherefore the ratio of FA to AD is given. But AF is the compound of BA and AC, for that CF is equal to AB: Therefore the ratio of the compound-line BAC to AD is given: The same demonstration will serve to shew that the rectangle contained under the compound-line BAC and ED is given also.

o 26, 29. 3.

p 22. 3.

q 4. 1.

PROP. XCV. Fig. 6.

If in the diameter BC of a circle ABC given in position, there be taken a given point D, and from that point D there be drawn a right-line DA, to the circumference of the circle. And if from the section of the said line there be drawn a right-line AE, perpendicular thereto, and through the point E where that perpendicular meets with the

X 2

the

the circumference, there be drawn a parallel EF, to the first line drawn AD, that point F in which the parallel meets with the diameter, is given; and the rectangle contained under the parallel-lines AD and EF is also given.

*Constr.* Let the right-line EF be prolonged to the point G, and let the right-line AG be drawn.

*Demonstr.* Forasmuch as the angle AEG is a right-angle, the right-line AG is the diameter of the circle. But BC is also the diameter: Therefore the point H is the center of the circle. Now the point D is given; and therefore (a) the line DH is given in magnitude. But seeing that AD is parallel to EG, and AH equal to GH; (b) DH is equal to FH, and AD to FG: (for the angles AHD and FHG (c) are equal, and DAH and FGH (d) are also equal.) But the line DH is given: Therefore FH is also given. But each of those lines DH and HF is also given in position, and the point H is given: Therefore (e) the point F is also given. And seeing that in the circle ABC given in position, is taken the given point F, and through the same is drawn the right-line EFG; the rectangle under EF and FG (f) is given. But FG is equal to AD. Therefore the rectangle comprehended under AD and EF is given. Which was to be demonstrated.

a 26. prop.

b 26. prop.

c 15. 1.

d 26. 1.

e 27. prop.

f 93. prop.

*The End of EUCLIDE'S DATA.*





A

## B R I E F T R E A T I S E

(Added by *FLUSSAS*)

O F

## Regular Solids:

**R**egular Solids are said to be composed and mix'd when each of them is transformed into other Solids, keeping still the form, number and inclination of the bases, which they before had to one another; some of which yet are transformed into mix'd Solids, and others into simple. Into mix'd, as a Dodecaedron and an Icosaedron, which are transformed or altered, if you divide their sides into two equal parts, and take away the solid angles subtended by plane superficial figures, made by the lines coupling those middle sections; for the Solid remaining after the taking away of those solid angles, is called an Icosidodecaedron. If you divide the sides of a Cube and of an

X 3

Octoedron



Octoedron into two equal parts, and couple the sections, the solid angles subtended by the plane superficies made by the coupling-lines, being taken away, there shall be left a Solid, which is called an Exoctoedron. So that both of a Dodecaedron, and also of an Icosaedron, the Solid which is made shall be called an Icosidodecaedron; and likewise the Solid made of a Cube, and also of an Octoedron, shall be called an Exoctoedron. But the other Solid, to wit, a Pyramis or Tetraedron, is transformed into a simple Solid, for if you divide into two equal parts each of the sides of the Pyramis, triangles described of the lines which couple the sections, and subtending and taking away the solid angles of the Pyramis, are equal and like unto the equilateral triangles left in each of the bases, of all which triangles is produced an Octoedron, viz. a simple, and not a composed Solid. For the Octoedron hath four bases, like in number, form, and mutual inclination with the bases of the Pyramis, and hath the other four bases with like situation opposite and parallel to the former. Wherefore the application of the Pyramis taken twice, maketh a simple Octoedron, as the other Solids make a mix'd compound Solid.

## DEFINITIONS.

## I.

*An Exoctoedron is a solid figure contained under six equal squares, and eight equilateral and equal triangles.*

## II.

*An Icosidodecaedron is a solid figure contained under twelve equilateral, equal, and equiangled Pentagons, and twenty equal and equilateral triangles.*

## PROBLEM I. Plate IX. Fig. 7.

*To describe an equilateral and equiangled Exoctoedron, and to contain it in a given sphere, and to prove that the diameter of the sphere is double to the side of the said Exoctoedron.*

*Constr.* Suppose a Sphere whose diameter let be AB, and about the diameter AB let there be described a square (a) and upon the square let there be described a Cube (b) which let be CDEFQTVR; and let the diameter thereof

a 6. 4.  
b 15. 13.

be  $QR$ , and the center  $S$ . Divide the sides of the Cube into two equal parts in the points  $G, H, I, K, L, M, N, O, P$ , &c. and join the middle sections by the right-lines  $IN, NO, OP, PI$ , and such like, which subtend the angles of the squares or bases of the Cube; and they are equal ( $c$ ) and contain right-angles, as the angle  $NIP$ . For the angle  $NID$ , which is at the base of the Isosceles triangle  $NDI$ , is the half of a right-angle, and so likewise is the opposite angle  $RIP$ . Wherefore the residue  $NIP$  is a right-angle, and so the rest. Wherefore  $NIPO$  is a square. And for the same reason shall the rest  $NMLK, KGHI$ , &c. inscribed in the bases of the Cube, be squares, and they shall be six in number, according to the number of the bases of the Cube. Again, forasmuch as the triangle  $RIN$  subtendeth the solid angle  $D$  of the Cube, and likewise the triangle  $KGL$  the solid angle  $C$ , and so the rest which subtend the right solid angles of the Cube, and these triangles are equal and equilateral (to wit) being made of equal sides, and they are the limits or borders of the squares, and the squares the limits or borders of them; as hath been before proved. Wherefore  $LMNOPHGK$  is an Exoctoedron by the definition, and is equilateral, for it is contained by equal subtendant lines; it is also equiangled, for every solid angle thereof is contained under two superficial angles of two squares, and two superficial angles of two equilateral triangles.

c 4. 1.

*Demonstr.* Forasmuch as the opposite sides and diameters of the bases of the Cube are parallels, the plane extended by the right-lines  $QT$  and  $VR$ , shall be a parallelogram. And because also in that plane lieth  $QR$ , the diameter of the Cube, and in the same plane also is the line  $MH$ , which divideth the said plane into two equal parts, and also coupleth the opposite angles of the Exoctoedron: this line  $MH$  therefore divideth the diameter into two equal parts ( $d$ ); and also divideth it self in the same point, which let be  $S$ , into two equal parts ( $e$ ). And by the same reason may we prove that the rest of the lines which join the opposite angles of the Exoctoedron, do in  $S$ , the center of the Cube, divide one another into two equal parts, for each of the angles of the Exoctoedron are set in each of the bases of the Cube. Wherefore making the center the point  $S$ , with the distance  $SH$  or  $SM$  describe a Sphere, and it shall touch every one of the angles equidistant from the point  $S$ .

d cor. 34. 1.

e 4. 1.

X 4

And



f 33. 3.

And because AB the diameter of the sphere given, is put equal to the diameter of the base of the cube, to wit, to the line RT, and the same line RT is equal to the line MH (*f*) which line MH coupling the opposite angles of the Exoctaedron, is drawn by the center. Wherefore it is the diameter of the Sphere given which containeth the Exoctaedron.

g 2. 6.

Lastly, forasmuch as in the triangle RFT, the line PO cuts the sides into two equal parts, it shall cut them proportionally with the bases, to wit, as FR is to FP, so shall RT be to PO. (*g*) But FR is double to FP by supposition: Wherefore RT, or the diameter HM, is also double to the line PO, the side of the Exoctaedron. Wherefore we have described, &c. Which was required to be done.

## PROBLEM II. Plate IX. Fig. 8.

*To describe an equilateral and equiangled Icosidodecaedron, and to comprehend it in a sphere given, and to prove that the diameter being divided in extreme and mean proportion, maketh the greater segment double to the side of the Icosidodecaedron.*

a 30. 6.

b 15. 13.

c 17. 13.

*Constr.* Suppose that the diameter of the sphere given be NL, (*a*) divide the line NL, in extreme and mean proportion in the point I, and the greater segment thereof let be NI, and upon the line NI describe a Cube; (*b*) and about this Cube let there be circumscribed a Dodecaedron; (*c*) and let the same be ABCDEFHKMO, and divide each of the sides into two equal parts in the points Q, R, S, T, V, X, Y, Z, P,  $\epsilon$ ,  $\delta$ , G, &c. and join the sections with right-lines, which shall subtend the angles of the Pentagons, as the lines PG, GV, VQ, QY, YR, RQ, VT, TX, XV, and so the rest.

d 4. 1.

*Demonstr.* Forasmuch as these lines subtend equal angles of the Pentagons, and those equal angles are contained by equal sides, to wit, by the half of the sides of the Pentagons; therefore those subtending lines are equal. (*d*) Wherefore the triangles GQV, YQR, and VXT, and the rest, which take away solid angles of the Dodecaedron, are equilateral.

Again, forasmuch as in every Pentagon are described five equal right-lines, coupling the middle sections of the sides, as are the lines QV, VT, TS, SR, and RQ, they describe a Pentagon in the plane of the Pentagon of the Dodecaedron. And the said Pentagon is contained in



in a circle, to wit, whose center is the center of a Pentagon of the Dodecaedron. For the lines drawn from that center to the angles of this Pentagon are equal, because they are perpendiculars upon the bases cut. (e) Wherefore the Pentagon QRSTV, is equiangled. (f) And by the same reason may the rest of the Pentagons described in the bases of the Dodecaedron, be proved equal and like.

e 12. 4.

f 11. 4.

Wherefore those Pentagons are twelve in number: And forasmuch as the equal and like triangles subtend and take away twenty solid angles of the Dodecaedron; therefore the said triangles shall be twenty in number. Wherefore we have described an Icosidodecaedron by the definition, which Icosidodecaedron is equilateral; for that all the sides of the triangles are equal and common with the Pentagons; and it is also equiangled: For each of the solid angles is made of two superficial angles of an equilateral Pentagon, and of two superficial angles of an equilateral triangle.

Now let us prove that it is contained in the given sphere whose diameter is NL. Forasmuch as perpendiculars drawn from the centers of the Dodecaedron, to the middle sections of its sides, are the halves of the lines which couple the opposite middle sections of the sides of the Dodecaedron, (g) which lines also (b) do in the center divide one another into two equal parts. Therefore right-lines drawn from that point to the angles of the Icosidodecaedron (which are set in those middle sections) are equal; which lines are thirty in number, according to the number of the sides of the Dodecaedron, for each of the angles of the Icosidodecaedron are set in the middle sections of each of the sides of the Dodecaedron. Wherefore making the center of the Dodecaedron, and the space, any one of the lines drawn from the center to the middle sections, describe a sphere, and it shall pass through all the angles of the Icosidodecaedron, and shall contain it.

g 3. cor. of

17. 13.

h idem.

And forasmuch as the diameter of this solid, is that right-line whose greater segment is the side of the Cube inscribed in the Dodecaedron, (i) which side is NI by supposition. Wherefore that solid is contained in the sphere-given, whose diameter is put to be the line NL.

i 4. cor.

17. 13.

Now let us prove that the great segment of the diameter is double to QV the side of the solid. Forasmuch as the sides of the triangle AEB are in the points Q and V divided into two equal parts, the lines QV and BE are parallels. (k) Wherefore as AE is to AV, so is EB to VQ.

k cor. 39. 1.

(l) But

l 2. 6.  
m 4. 6.  
n 2. cor. of  
17. 13.

(*l*) But the line AE is double to the line AV. Wherefore the line BE is double to the line QV. (*m*) Now the line BE is equal to NI, or to the side of the Cube; (*n*) which line NI is the greater segment of the diameter NL. Wherefore the greater segment of the diameter given is double to the side of the Icosidodecaedron inscribed in the given sphere. Wherefore we have described, &c. Which was required to be done.

#### ADVERTISEMENT.

To the understanding of the nature of this Icosidodecaedron, you must well conceive the passions and properties of both these solids, of whose bases it consisteth, to wit, of the Icosaedron and of the Dodecaedron. And altho' in it the bases are placed oppositely, yet have they to one another one and the same inclination. By reason whereof there lie hidden in it the actions and passions of the other regular Solids. And I would have thought it not impertinent to the purpose to have given the inscriptions and circumscriptions of this Solid, if want of time had not hindered. But that the Reader may the better attain to the understanding thereof, I have here following briefly shewn, how it may in or about every one of the five regular Solids be inscribed or circumscribed; by the help whereof he may, with small labour, or rather none at all, having well poised and considered the Demonstrations appertaining to the foresaid five regular Solids, demonstrate both the inscription of the said Solids in it, and the Inscription of it in the said Solids.

#### *Of the Inscriptions and Circumscriptions of an Icosidodecaedron.*

An Icosidodecaedron may contain the other five regular bodies. For it will receive the angles of a Dodecaedron in the centers of the triangles which subtend the solid angles of the Dodecaedron, which solid angles are twenty in number, and are placed in the same order in which the solid angles of the Dodecaedron, taken away, or subtended by them, are. And for that reason it shall receive a Cube and a Pyramis contained in the Dodecaedron, when as the angles of the one are set in the angles of the other.

An Icosidodecaedron receiveth an Octoedron, in the angles cutting the six opposite sections of the Dodecaedron, even as if it were a simple Dodecaedron.

And



And it containeth an Icosaedron, placing the twelve angles of the Icosaedron in the same centers of the twelve Pentagons.

It may also by the same reason be inscribed in each of the five regular bodies, to wit, in a Pyramis, if you place four triangular bases concentrical with four bases of the Pyramis, after the same manner that you inscribed an Icosaedron in a Pyramis; so likewise may it be inscribed in an Octoedron, if you make eight bases thereof concentrical with the eight bases of the Octoedron. It shall also be inscribed in a Cube, if you place the angles which receive the Octoedron in it, in the centers of the bases of the Cube. Again, you shall inscribe it in an Icosaedron, when the triangles formed of the Pentagon bases, are concentrical with the triangles which make a solid angle of the Icosaedron.

Lastly, it shall be inscribed in a Dodecaedron, if you place each of the angles thereof in the middle sections of the sides of the Dodecaedron, according to the order of its construction.

The opposite plain superficies also of this solid are parallels. For the opposite solid angles are subtended by parallel plain superficies, as well in the angles of the Dodecaedron subtended by triangles, as in the angles of the Icosaedron subtended by Pentagons, which thing may easily be demonstrated. Moreover, in this solid are infinite properties and passions, springing from the solids whereof it is composed.

Wherefore it is manifest, that a Dodecaedron and an Icosaedron mixed, are transformed into one and the self same solid of an Icosidodecaedron. A Cube also and an Octoedron are mixed and altered into another solid, to wit, into one and the same Exoctoedron. But a Pyramis is transformed into a simple and perfect solid, to wit, into an Octoedron.

If we will frame these two solids joined together into one solid, this only must we observe.

In the Pentagon of a Dodecaedron inscribe a like Pentagon, and let its angle be set in the middle sections of the Pentagon circumscribed, and then upon the said Pentagon inscribed, let there be set a solid angle of an Icosaedron, and so observe the same order in each of the bases of the Dodecaedron, and the solid angles of the Icosaedron, set upon these Pentagons, shall produce a solid consisting of the whole Dodecaedron, and whole Icosaedron. In like manner, if in every base of the Icosaedron



saedron, the sides being divided into two equal parts, be inscribed an equilateral triangle, and upon each of those equilateral triangles be set a solid angle of a Dodecaedron, there shall be produced the same solid consisting of the whole Icosaedron, and of the whole Dodecaedron.

And after the same order, if in the bases of a Cube be inscribed squares subtending the solid angles of an Octoedron, or if in the bases of an Octoedron be inscribed equilateral triangles subtending the solid angles of a Cube, there shall be produced a solid consisting of either of the whole solids, to wit, of the whole Cube, and of the whole Octoedron.

But equilateral triangles inscribed in the bases of a Pyramis, having their angles set in the middle sections of the sides of the Pyramis, and the solid angles of a Pyramis, set upon the said equilateral triangles, there shall be produced a solid consisting of two equal and like Pyramids.

And now if in these solids thus composed, you take away the solid angle, there shall be restored again the first composed solids, to wit, the solid angles taken away from a Dodecaedron and an Icosaedron composed into one, there shall be left an Icosidodecaedron, the solid angles taken away from a Cube and an Octoedron composed into one solid, there shall be left an Exoctoedron. Moreover, the solid angles taken away from two Pyramids composed into one solid, there shall be left an Octoedron.

*Of the nature of a trilateral and equilateral Pyramis,*

1. A trilateral equilateral Pyramis is divided into two equal parts, by three equal squares, which in the center of the Pyramis cut one another into two equal parts, and perpendicularly, and whose angles are set in the middle sections of the sides of the Pyramis.

2. From a Pyramis are taken away four Pyramids like unto the whole, which utterly take away the sides of the Pyramis, and that which is left is an Octoedron, inscribed in the Pyramis, in which all the solids inscribed in the Pyramis are contained.

3. A perpendicular drawn from the angle of the Pyramis to the base, is double to the diameter of the Cube inscribed in it.

4. And a right-line joining the middle sections of the opposite sides of the Pyramis is triple to the side of the same Cube.

5. The

1. The  
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segment.

5. The side also of a Pyramis is triple to the diameter of the base of the Cube.

6. Wherefore the same side of the Pyramis is in power double to the right-line which joins the middle section of the opposite sides.

7. And it is in power sesquialter to the perpendicular which is drawn from the angle to the base.

8. Wherefore the perpendicular is in power sesquiter-tia to the line which joins the middle sections of the opposite sides.

9. A Pyramis and an Octoedron inscribed in it, also an Icosaedron inscribed in the same Octoedron, contain one and the same sphere.

*Of the nature of an Octoedron.*

1. Four perpendiculars of an Octoedron, drawn in four bases thereof from two opposite angles of the said Octoedron, and joined together by those four bases, describe a Rhombus, or Diamond figure; one of whose diameters is in power double to the other diameter.

2. For it hath the same proportion that the diameter of the Octoedron hath to the side of the Octoedron.

3. An Octoedron and an Icosaedron inscribed in it, do contain one and the same sphere.

4. The diameter of the solid of the Octoedron is in power sesquialter to the diameter of the circle which containeth the base, and is in power duple superbipartiens tertias (that is, as 8 to 3,) to the perpendicular or side of the foresaid Rhombus; and moreover is in length triple to the line which joins the centers of the next bases.

5. The angle of the inclination of the bases of the Octoedron, doth, with the angle of the inclination of the bases of the Pyramis, make angles equal to two right-angles.

*Of the nature of a Cube.*

1. The diameter of a Cube is in power sesquialter to the diameter of his base.

2. And is in power triple to his side.

3. And unto the line which joins the centers of the next bases, it is in power sextuple.

4. Again, the side of the Cube, is to the side of the Icosaedron inscribed in it, as the whole is to the greater segment.

5. Unto

5. Unto the side of the Dodecaedron, it is as the whole is to the lesser segment.

6. Unto the side of the Octoedron it is in power duple.

7. Unto the side of the Pyramis it is in power subduple.

8. Again, the Cube is triple to the Pyramis, but to the Cube the Dodecaedron is in a manner double. Wherefore the same Dodecaedron is in a manner sextuple to the said Pyramis.

*Of the nature of the Icosaedron.*

1. Five triangles of an Icosaedron, make a solid angle, the bases of which triangles make a Pentagon. If therefore from the opposite bases of the Icosaedron be taken the other Pentagon by them described, these Pentagons shall in such sort cut the diameter of the Icosaedron which joins the foresaid opposite angles, that that part which is contained between the planes of these two Pentagons shall be the greater segment, and the residue which is drawn from the plane to the angle, shall be the lesser segment.

2. If the opposite angles of two bases joined together, be coupled by a right-line, the greater segment of that right-line is the side of the Icosaedron.

3. A line drawn from the center of the Icosaedron to the angles, is in power quintuple to half that line which is taken between the Pentagons, or of the of half that line which is drawn from the center of the circle which containeth the foresaid Pentagon, which two lines are therefore equal.

4. The side of the Icosaedron containeth in power either of them, and also the lesser segment, to wit, the line which falleth from the solid angle to the Pentagon.

5. The diameter of the Icosaedron containeth in power the whole line, which coupleth the opposite angles of the bases joined together, and the greater segment thereof, to wit, the side of the Icosaedron.

6. The diameter also is in power quintuple to the line which was taken between the Pentagons, or to the line which is drawn from the center to the circumference of the circle which containeth the Pentagon composed of the sides of the Icosaedron.

7. The



7. The dimetient containeth in power the right-line which coupleth the centers of the opposite bases of the Icosaedron, and the diameter of the circle which containeth the base.

8. Again, the said dimetient containeth in power the diameter of the circle which containeth the Pentagon, and also the line which is drawn from the center of the same circle to the circumference: that is, it is quintuple to the line drawn from the center to the circumference.

9. The line which coupleth the centers of the opposite bases containeth in power the line which coupleth the centers of the next bases, and also the rest of that line of which the side of the Cube inscribed in the Icosaedron is the greater segment.

10. The line which coupleth the middle sections of the opposite sides, is triple to the side of the Dodecaedron inscribed in it,

11. Wherefore, if the side of the Icosaedron and the greater segment thereof be made one line, the third part of the whole is the side of the Dodecaedron inscribed in the Icosaedron.

*Of the Dodecaedron.*

1. The diameter of a Dodecaedron containeth in power the side of the Dodecaedron, and also that right-line to which the side of the Dodecaedron is the lesser segment, and the side of the Cube inscribed in it is the greater segment, which line is that which subtendeth the angle of the inclination of the bases, contained under two perpendiculars of the bases of the Dodecaedron.

2. If there be taken two bases of the Dodecaedron, distant from one another by the length of one of the sides, a right-line coupling their centers being divided in extreme and mean proportion; maketh the greater segment the right-line which coupleth the centers of the next bases.

3. If by the centers of five bases set upon one base, be drawn a plain superficies, and by the centers of the bases which are set upon the opposite base, be drawn also a plain superficies, and then be drawn a right-line, coupling the centers of the opposite bases, that right-line is so cut, that each of his parts set without the plain superficies, is the greater segment of that part which is contained between the planes.

4. The

4. The side of the Dodecaedron is the greater segment of the line which subtendeth the angle of the Pentagon.

5. A perpendicular-line drawn from the center of the Dodecaedron to one of the bases, is in power quintuple to half the line which is between the planes.

6. And therefore the whole line which coupleth the centers of the opposite bases is in power quintuple to the whole line which is between the said planes.

7. The line which subtendeth the angle of the base of the Dodecaedron, together with the side of the base, are in power quintuple to the line which is drawn from the center of the circle which containeth the base, to the circumference.

8. A section of a sphere containing three bases of the Dodecaedron, taketh a third part of the diameter of the said sphere,

9. The side of the Dodecaedron and the line which subtendeth the angle of the Pentagon, are equal to the right-line which coupleth the middle sections of the opposite sides of the Dodecaedron.

THE

T H E  
T H E O R E M S  
O F  
A R C H I M E D E S.

Concerning the Sphere and Cylinder,  
investigated by the *Method of indivi-*  
*sibles*, and briefly demonstrated, by the  
Reverend and Learned Dr. I s A A C  
B A R R O W.

**T**HE main Design of *Archimedes* in his *Treatise of the Sphere and Cylinder*, is to resolve these four Problems.

1. To find the proportion of the superficies of a sphere to any determinate circle; or to find a circle equal to the superficies of a given sphere.
2. To find the proportion of the superficies of any segment of a sphere to any determined circle; or to find a circle equal to the superficies of any assigned segment.
3. To find the proportion of the sphere it self (or of its solid content) to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given sphere.



4. To find the proportion of a segment of a sphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given segment.

These four Problems *Archimedes* prosecutes separately, and lays down Theorems immediately subservient to their solution; but we reduce them to two: For since an Hemisphere is the segment of a sphere, and the method of finding out its relations, in respect to the superficies and solid content, is comprehended in the general method of investigating the proportion of the segments: And from the superficies and solid content of an Hemisphere already found, the double of them, (that is the superficies and content of the whole sphere) is at the same time given. And indeed 'tis superfluous and foreign from the Laws of good Method, to investigate their relations distinctly and separately; so that if it were not a crime, I might on this account blame even *Archimedes* himself.

The whole matter therefore is reduc'd to these two Problems.

1. To find the proportion of the superficies of any segment of a sphere to a determinate circle; or to find a circle equal to the superficies of a given segment.

2. To find the proportion of the solidity of any segment of a sphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to an assign'd segment of a sphere.

I shall resolve these Problems by another much easier and shorter method: In which the order being inverted, first, I shall seek the solidity of a segment, and from thence deduce its superficies; a thing which is in my judgment well worth observing, and perform'd (as I know of) by none.

First therefore, for finding the solidity of a segment, I shall lay down two, commonly known and receiv'd, Suppositions, *viz.*

1. That a series of magnitudes proceeding in Arithmetical Progression from nothing (inclusive) or whose common difference is equal to the least magnitude, is subduple of as many quantities equal to the greatest: (i. e. subduple of the product of the greatest term and number of terms:) So that if the sum of the terms be called  $\Sigma$ , the greatest term  $g$ , and the number of terms  $n$ , then will

$$\Sigma = \frac{n g}{2}, \text{ or } 2 \Sigma = n g.$$

The

The truth of this Proposition will easily appear by expressing the series twice, and inverting the order;

$0, a, 2a, 3a, 4a.$

$4a, 3a, 2a, a, 0.$

For so the difference always being equal to the least quantity, 'twill be evident that each two correspondent terms taken together are equal to the greatest term; and also, that the series taken twice is equal to the greatest term repeated as many times as there are terms, *i. e.* the last term drawn into the number of terms.

We have in a *triangle* a very clear and easy example of this most useful Proposition, which is prov'd hence, *to be half a parallelogram having the same altitude, and standing on the same base.*

Suppose the altitude AE (Plate IX. Fig. 9.) of the triangle AEZ to be divided into parts indefinitely many and small, AB, BC, CD, DE, and parallels BZ, CZ, DZ, EZ, drawn thro' the points of Division; all these proceed from nothing in an Arithmetical progression, and consequently the sum of them all, (that is, the triangle AEZ) is subduple of the greatest EZ drawn into the altitude AE, by which the sum of the terms is express'd, that is subduple of the Parallelogram EY, whose base is EZ, and altitude AE.

But the illustration of the Rule will conduce more to our design by inferring hence, *That a circle is equal to half of the radius drawn into the circumference*, after this manner. Conceive a circle to consist of as many concentric Peripheries as there are points or equal parts indefinitely many and small in the radius. These Peripheries, as well as their radii, proceed from the center or nothing in an Arithmetical progression; and therefore their sum, that is, the whole circle is equal to half the greatest (or extreme circumference) drawn into the number of terms, that is, the radius.

Y 2

After

After the same manner we may suppose the sector AEZ (Fig. 10.) to consist of as many concentric Arcs BZ, CZ, DZ, EZ, as there are points (or equal parts indefinitely small) in the radius AE, which Arcs, as their radii, proceeding from a point or nothing in an Arithmetical progression, the sector also will be equal to half the radius drawn into the extreme Arc EZ. Which may be made evident also after this manner: Let us suppose the right line EY to be perpendicular to the radius AE, draw the right-line AY, and from the points B, C, D, of division in the radius, draw BY, CY, DY, parallel to EY, and terminated at AY. Because  $EY : DY :: \text{rad. AE} : \text{rad. AD} :: \text{Arc. EZ} : \text{Arc. DZ}$ , and  $EY = EZ$ , then will  $DY = \text{Arc. DZ}$ ; and in like manner will  $CY = CZ$ , and  $BY = BZ$ . Whence the triangle AEY Will be = to the sector

$$\text{AEZ, that is, } \frac{AE \times EY}{2} = \frac{AE \times EZ}{2} = \text{sector AEZ. By}$$

this means we collect that celebrated Theorem of Archimedes, that a circle is equal to a triangle whose base is equal to the radius, and altitude equal to the periphery of the circle; and that without any inscription or circumscription of figures, by only supposing that the Area or Superficies of the circle consists of infinitely many concentric Peripheries. Which method of Indivisibles, (now first of all known to me) seems no less evident (nay more evident) and perhaps less fallacious than that wherein planes are supposed to consist of parallel right-lines, and solids of parallel planes; as hereafter shall be evident, when we shall collect, by this method, the proportions of spheric and cylindric superficies to one another, by knowing the solid content; and on the other hand, the solid content, by knowing the superficies, with admirable facility, and full satisfaction in those things which are with difficulty obtained by pure Geometry.

2. Let us suppose a series of quantities to proceed from 0 (inclusive) in a duplicate Arithmetic proportion, that is, 0, 1, 4, 9, 16, &c. the squares of numbers in a simple Arithmetic progression, 0, 1, 2, 3, 4, &c. And the triple of this series will always exceed the greatest term multiplied by the number of terms; but the number of terms increasing, the proportion continually approximates, till at last it comes to an equality, when the number of terms is increased in infinitum.



$$\begin{array}{rcl}
 3 \times 0 + 1 & = & 3. \quad \underline{3} \\
 2 \times 1 & = & 2. \quad \underline{2} \\
 3 \times 0 + 1 + 4 & = & 15. \quad \underline{15} \quad 5 \\
 3 \times 4 & = & 12. \quad \underline{12} \quad 4 \\
 3 \times 0 + 1 + 4 + 9 & = & 42. \quad \underline{42} \quad 7 \\
 4 \times 9 & = & 36. \quad \underline{36} \quad 6 \\
 3 \times 0 + 1 + 4 + 9 + 16 & = & 90. \quad \underline{90} \quad 9 \\
 5 \times 16 & = & 80. \quad \underline{80} \quad 8 \\
 3 \times 0 + 1 + 4 + 9 + 16 + 25 & = & 165. \quad \underline{165} \quad 11 \\
 6 \times 25 & = & 150. \quad \underline{150} \quad 10
 \end{array}$$

As for example, if the terms are two, the triple of the terms will be to the greatest term drawn into the number of terms as 3 to 2; if there be three terms as 5 to 4; if four, as 7 to 6; if five, as 9 to 8, and so continually: So that the antecedents of these proportions always mutually exceed one another by the number 2, and so every antecedent its consequent by 1. Whence it is evident that by how much the greater the number of terms is, by so much the more the proportion tends to equality. So 100 to 99 is less distant from the proportion of equality than 10 to 9. From hence, supposing the number of terms infinite (or infinitely great,) the triple of quantities proceeding thus in a duplicate proportion (or as the squares of the numbers, 0, 1, 2, 3, 4, &c.) will be equal to as many quantities equal to the greatest term.

The same, as to the substance of it, is laid down by *Archimedes* in his *Book of Spirals*, as the Foundation of many Argumentations, in that, and other Books, and is well demonstrated by our learned Country-man *Dr. Wallis*: However, I thought fit to illustrate the matter by this method, as being not unworthy our Consideration, and very perspicuous and intelligible in this, that 'tis free from Fractions: And by the way 'tis observ'd, that from hence we may easily find the proportion of a series triple to as many terms equal to the greatest, viz. as twice the number of terms less one, to twice the number of terms less two. So that if the number of the terms be 6, the proportion of a series triple to as many terms equal to the greatest will be as 11.

It will be a very easy and apt Illustration of this Rule, if we infer hence, *That a Cone is subtriple of a Cylinder, having an equal base and altitude.* For let us suppose the altitude AE (Fig. 11.) of the Cone ZY to be divided into equal and indefinitely number of parts, by as many parallel right-lines ZY, and the lines ZY will be as the numbers 1, 2, 3, 4, &c. and the squares or circles constituted upon the diameters ZY, as 1, 4, 9, 16, &c. whence all those circles, or the whole Cone AZY (made up of the same) will be subtriple of as many circles equal to the greatest, constituted on the greatest diameter ZEY, that is, subtriple of a cylinder whose base is AEY, and altitude AE.

There occur two other most apt examples of this Rule, viz. by inferring, *That the complement of a Semiparabola is subtriple of a parallelogram having the same base and height; as also, That the space comprehended by the Spiral and Radius is subtriple of the circle in which the Spiral is generated:* But of these in another place. Wherefore to go on with what we began, these two Rules being supposed; let us conceive ZAY (Fig. 12.) to be a segment of a sphere, X its center, AT its diameter, and ZAYT a great circle passing thro' the vertex, and the part AE of the Axe to be divided into an indefinitely number of equal parts; and let us imagine parallel-lines to be drawn thro' the points of division generating circles in the sphere, whose Radii let be BZ, CZ, DZ, and diameters ZY. I suppose the segment of a sphere to consist of all these parallel circles, whose number is as great as that of the points, or equal indefinitely small parts in the Axe AE, according to the known *Method of Indivisibles*.

But nor for brevity's sake, let the diameter AT be called  $d$ , and the radius of the sphere  $r$ , (if need be) and the Axe AE, by which the number of terms is express'd, call  $n$ , and one of the equal parts  $a$ ; which being settled, 'tis evident, (*by the Elements*) that  $BZ^2 = AB \times BT = a \times d - a = ad - a^2$ , and in like manner  $CZ^2 = AC \times CT = 2a \times d - 2a = 2ad - 4a^2$ , and by the same reasoning  $DZ^2 = AD \times DT = 3ad - 9a^2$ , and  $EZ^2 = AE \times ET = 4ad - 16a^2$ , &c. that is, that the

the squares of the radii of the circles ZY are to one another as the rectangles,  $ad$ ,  $2ad$ ,  $3ad$ ,  $4ad$ , &c. (which proceed in an Arithmetical Progression from 0) less by the square  $a^2$ ,  $4a^2$ ,  $9a^2$ ,  $16a^2$ , &c. which go on as the squares of the number, 1, 2, 3, 4, &c. But by our first Rule, all the Rectangles 0,  $ad$ ,  $2ad$ ,  $3ad$ ,  $4ad$ , &c. are equal to half as many terms equal to the greatest AE

$\frac{nd \times n}{2}$   $\times$  AT or  $nd$ , that is,  $= \frac{nd^2}{2}$ .

Moreover, by our second Rule, all the squares 0,  $a^2$ ,  $4a^2$ ,  $9a^2$ ,  $16a^2$ , &c. taken together, are equal to a third part of as many terms equal to the greatest  $AE^2$  or  $n^2$ , that is,  $= \frac{n^3}{3}$ .

Wherefore all the squares described upon the radii BZ, CZ, DZ, EZ, conjunctly, are equal to the difference  $\frac{nd^2}{2} - \frac{n^3}{3}$ , (or the terms being reduc'd to the

same denomination,)  $\frac{3ndn - 2nnn}{6}$  and their quadruple,

that is, all the squares described upon the diameters ZY, are equal to  $\frac{12ndn - 8n^3}{6}$  or  $\frac{6ndn - 4n^3}{3}$  whence a

segment of a sphere is equal to a Cylinder, the diameter of whose base is the side of a square equal to  $6nd - 4n^2$ , and altitude is  $\frac{1}{3}n$ ; or to a Cone having the same base, but the altitude  $n$ , or which is all one, having a base

whose radius is  $\sqrt{\frac{6nd - 4n^2}{4}}$  or  $\sqrt{\frac{3}{2}nd - n^2}$ , and al-

titude  $n$  as before. Which Cone we may change into a Cone upon the same base ZY with the segment ZAY, by saying, as  $ZE^2$  (i. e.  $dn - n^2$ ) to  $\frac{1}{2}nd - n^2$  or (both terms being divided by  $n$ ) as  $d - n$  to  $\frac{1}{2}d - n$ , so is reciprocally  $n$  to the altitude of the Cone sought: Or in the figure by making, as TE to TE + XA, so is EA to ES. For ES will be the altitude of the Cone ZSY equal to the segment of the sphere ZAY. Which is that noted Theorem of Archimedes, demonstrated by him with so much labour and prolixity.



Hence, if the given segment be a *Hemisphere*, and so  $n = \frac{1}{2} d$  or  $r$ , then  $d$  or  $2r$  will be the altitude of a *Cone*, which having a base equal to the base of the *Hemisphere* (or to the greatest circle in the sphere) will be equal to the *Hemisphere*. And a *Cone* whose base is double of the greatest circle, and the altitude  $2r$ , or the *Cylinder* whose bases is  $\frac{1}{2}$  of the greatest circle, and altitude  $2r$  will be equal to the whole *Sphere*. Whence the whole *Sphere* is  $\frac{2}{3}$  of a *Cylinder*, the diameter of whose base is  $2r$ , and the altitude also  $2r$ . And this is the chief Theorem of *Archimedes*, viz. *That a sphere is subsequalter or  $\frac{2}{3}$  of that Cylinder, whose Altitude and Diameter of the base is equal to the Diameter of the Sphere.*

Furthermore, not to pass over any thing in our Author which seems to be to our purpose :

If to the sum first found, representing a segment, viz.  $\frac{6ndn-4nnn}{3}$ , we add  $\frac{2d^2n-6dn^2+4n^3}{3}$   $\frac{4dn-4n^2}{3}$   $\times \frac{d-2n}{2} = \frac{4}{3} ZE^2 \times XE$  representing the Cone ZXY,

the aggregate  $\frac{2}{3} d d n$  will represent the Sector of the Sphere ZXYA, which for that reason will be equal to a *Cylinder*, the diameter of whose base is  $\sqrt{dn}$ , and the altitude  $\frac{2}{3} d$ , or to a *Cone*, the diameter of whose base is  $\sqrt{dn}$ , and the altitude  $2d$ , or also to a *Cone*, the Radius of whose base is  $\sqrt{dn}$ , and the altitude  $\frac{1}{2} d = r$  (it being reciprocally as  $4dn : dn :: 2d : \frac{1}{2} d$ .) that is, to a *Cone*, the Radius of whose base is the Line AZ, drawn from the vertex to the circumference of the base of the segment, (for  $AZ^2 = TA \times AE = dn$ .) and the altitude  $r$ . And this is the next famous Theorem of *Archimedes*, concerning the solidity of the sector of the Sphere, viz. *That the sector of a sphere is equal to a Cone, whose base is a circle described by a Radius equal to a line drawn from the vertex to the circumference of the base of the segment, and whose altitude is equal to the Radius of the sphere.*

And thus I think I have compleated that which belongs to the solidity of a sphere, and its parts, with sufficient brevity and perspicuity. From hence we shall deduce the Resolution of the other Problem, which I proposed concerning the surface of the segment of a sphere; and then of the whole sphere. To obtain this, as we supposed before, a *Circle* to consist of concentric Peripheries,

Peripheries, and the *Sector of a Circle* of concentric Arcs, (in the number of which, the greatest, and the least, or a point is reckon'd: So now we suppose spheres to consist of concentric spherical superficies, and the *Sector of Spheres* of like concentric superficies; as for example, the sector of the sphere ZAE (*Fig. 13.*) of the superficies BZ, CZ, DZ, EZ, &c.) which supposition indeed seems so easy and natural, that in my judgment 'tis sufficient only to propose it; neither is a further explanation wanting to gain an assent to it.

2. We suppose these spherical superficies to be in a duplicate Ratio of the Radius of the spheres: This is the common affection of all like superficies, and it seems to agree very well with the superficies of spheres, because they appear to be most uniform and similar. But this Supposition might easily be evinc'd and establish'd by the same sort of arguing, as spheres are proved to be in triplicate proportion to their Diameters or Radii; or might have been join'd as a *Corollary* to *Prop. 17. and 18. Elem. 12.* where the superficies of like Polygons are suppos'd to be inscribed in spheres, having as well the superficies in a duplicate, as the solidity in a triplicate Ratio of the Diameters of the Spheres. These things being premis'd let us suppose AE a Radius, or the side of the Sector of a Sphere EAZ, to be divided into equal and indefinitely many small parts, and the sector AEZ to consist of these spherical superficies BZ, CZ, DZ, EZ, it will be evident that all those superficies in the Progression are as the squares of the Radii, that is, as  $\overline{AB^2}$ ,  $\overline{AC^2}$ ,  $\overline{AD^2}$ ,  $\overline{AE^2}$ , &c. or as the squares of the numbers 1, 2, 3, 4, &c. whence by our second Rule, the sum of all these superficies, that is, the sector AEZ, will be  $\frac{1}{3}$  of as many superficies equal to the greatest EZ, that is,  $\frac{1}{3}$  of the greatest EZ, drawn into  $r$  the number of terms. Whence a sector is equal to a Cylinder, whose base is  $\frac{1}{3}$  of the greatest or extreme superficies of the sector, and whose altitude is  $r$ : Or to a Cone whose base is equal to the superficies of the sector, and its altitude  $r$ , which is the last of *Lib. 1.* but we just now prov'd that a sector is equal to a Cone whose altitude is  $r$ , and base a circle, describ'd by the Radius YB, drawn from the vertex of the segment EYZ to the circumference of the base. Wherefore a Cone, whose altitude is  $r$ , and base equal to the superficies of the sector, is equal to a Cone of the same altitude, whose base is a circle describ'd by the Radius YE,

And



And so the superficies of the sector EYZ is equal to a circle describ'd by the Radius YE. Which certainly is the principal Theorem of all those that occur in the Books of *Archimedes*, nor is there found a more excellent one in all Geometry; viz. *That the superficies of any segment of a sphere is equal to a circle whose Radius is a right-line drawn from the vertex of the segment to the circumference of the base: And hence, that the superficies of an Hemisphere is double to the base, or equal to two great circles of the sphere.*

For in this Case  $\overline{YE^2} = \overline{AZ^2} + \overline{AY^2} = 2 \overline{AE^2}$ , and consequently a circle described by the Radius YE (*Fig. 14.*) is equal to two circles describ'd by the Radius AE. Whence also, *the superficies of the whole sphere is quadruple, a circle having the same Radius with the sphere, that is, quadruple the greatest circle in the sphere; and equal to a circle whose Radius is the diameter of the sphere.* From hence it follows, *that the superficies of a sphere is equal to the superficies of a Cylinder of the same height and breadth; for the superficies of that Cylinder is quadruple to the base as we shall shew hereafter. And these are the most noted Theorems of Archimedes.* Nay, from hence all those things follow, which he has written concerning the superficies of spheres, and their segments. So that from these few and easy Suppositions, I have demonstrated whatever seems to be of any Note in the Books of *the Sphere and Cylinder*.

I will only add, that after by the method of *Archimedes*, (for I think scarce any other can be invented, besides ours, for finding the solidity) the superficies of segments are found equal to the circle described by the Radii YE; hence it will plainly follow, that the superficies of spheres, and thence of like sectors are in a duplicate ratio of the Radii of the spheres; and consequently from the superficies thus found, the contents of segments, and of whole spheres may be mutually deduced, and that very clearly and expeditiously, after this manner. Because in the sector EAZ (*Fig. 13.*) the superficies BZ, CZ, DZ, EZ, proceed as the squares describ'd upon AB, AC, AD, AE, that is, as 1, 4, 9, 16, &c. the whole sector will be equal to  $\frac{1}{3}$  of as many superficies equal to the greatest EZ, or  $\frac{1}{3} EZ \times r$ , that is, to a Cylinder whose base is  $\frac{1}{3} EZ$ , and altitude  $r$ , or to a Cone whose base is EZ, and altitude  $r$ . But EZ is supposed equal to a circle whose Radius is YE, wherefore the sector EAZ



is equal to a Cone whose base is a circle described by the Radius YZ and altitude  $r$ : Which is *Archimedes's* universal Theorem for the contents of *Sectors*. Whence if from this the Cone ZAE standing on the base of the segment EYZ, and having the vertex at the center of the sphere A, be subducted, you'll have that segment EYZ. But when the sector EYZ is a Hemisphere, there will be no such Cone to be subducted; and for that reason a Cylinder whose base is  $\frac{2}{3}$  EZ, and altitude  $r$ , or the Cone whose base is  $2$  EZ, and altitude likewise  $r$ , will be equal to the whole sphere. But the Superficies of the Hemisphere EZ, is proved to be equal to two of the greatest circles in the sphere, whence the whole sphere is given. This is *Archimedes's* first and principal Theorem, for the content of a sphere; whence 'tis easily deduced, that a sphere is  $\frac{2}{3}$  of a circumscrib'd Cylinder, that is, of a Cylinder whose altitude and diameter of its base is equal to the diameter of the sphere.

The Doctrine of our author [*Archimedes*] seems to make against, and subvert the new and celebrated *Method of indivisibles*, and is press'd to that end by *Tacquet*; for instance, (*Prop. 2. lib. 2. Cylindr.*) For the usual process of that method seems to exhibit the dimensions of the superficies of a Cone, (as also of a sphere, and of other Curves) different enough from what our Author and others have demonstrated: As for example, let us suppose ABCD (*Fig. 15.*) a right Cone, whose Axe is AX, base BCD, and plane  $bxd$  drawn, at pleasure, parallel to the base BCD. And since, as *Diam. BD* : *Periph. BCD* : : *Diam. bd* : *Periph. bxd*, and so every where it will be (according to the *Method of Indivisibles*, and by 12. 5.) as *Diam. BD*, to *Periph. BCD*, so is the triangle ABD, consisting of those parallel Diameters, to the Conic Superficies ABCD, consisting of those Peripheries, i. e. *Diam. BD* : *Periph. BCD* : :  $AX \times BD$   $AX \times Periph. BCD$

\_\_\_\_\_ : \_\_\_\_\_ , Whence

$$\frac{AX \times Periph. BCD}{2} \quad 2$$

\_\_\_\_\_ will be equal to the superficies of

the Cone; which is false and contrary to what was demonstrated just now. For we demonstrated that the

$$AB \times Periph. BCD$$

superficies of the Cone was \_\_\_\_\_.

In answering this Objection, we say, that the *Method of Indivisibles*, in the speculation of Perimeters, and of Curve Surfaces, proceeds otherwise than in the speculation of plane Surfaces and solid Contents. It does indeed suppose that the Area of plane Figures consists as it were of parallel right-lines, and the contents of solids of parallel Planes, and that their number may be express'd by the altitude of the Figures: But it by no means supposes, that the Perimeters of plane figures consist of points, or the superficies of solids of lines, the number of which may be express'd by the altitude of the figure. As for example, altho' the triangle ABD (in the last figure) consists of lines parallel to BD, the number of which is expressed by the number of points in the perpendicular AX, that is, by the length of the perpendicular: Yet it would be absurd to suppose that the line AB consists of points, whose number may be express'd by the number of points in a less line AX. For altho' the right-lines *b d* drawn thro' each infinitely small part of AX, divide AB into as many infinitely small parts, yet those parts are not of the same Denomination or Quality with the parts of AX, but somewhat greater than them; so that if the parts of AX be look'd upon as points, the parts of AB are not to be called points, but greater than points; and on the contrary, if the parts of AB be called points, the parts of AX are to be look'd upon as less than points, if it be lawful to speak so. For the points which are treated of in the *Method of Indivisibles* are not absolutely points, but indefinitely small parts, which usurp the name of points, because of their affinity to them. Since therefore points don't admit of greater and less, the name of points is not at the same time to be attributed to the parts of different magnitudes; consequently, tho' the number of the greater parts of AB may be express'd by the number of the lesser parts of AX, yet the number of points in AB can no ways be expressed by the number of points in AX, (that is, by the number of parts in AX, equal to the number of parts in AB, which are called points.) The line AB has as many points as there are in it self alone, or another line equal to it self, nor can it be determined by any other measure. After the same manner, this method don't suppose the conic Surface ABCD to consist of as many parallel circumferences perpetually



perpetually increasing from the vertex A, or decreasing from the base BD, as there are points in the Axe AX, but rather of as many thus increasing or decreasing as there are points in the side AB. For in the Revolution of the line AB about the Axis AX, (whereby the superficies of the Cone is generated) every point in the line AB produces a circumference, and consequently more circumferences are produced than the points contained in the Axis AX. Therefore if you would extend the *Method of Indivisibles* to the superficies of solids, and suppose those superficies to consist of parallel-lines, you ought not to compute this by the parallel Areas constituting the solids, that is, not to number those Areas by the altitude of the solid, but by other lines agreeable to the condition of each figure. Which lines, in figures that are not irregular, may easily be determined: For instance, in the equilateral Pyramid ABCD, (*Fig. 16.*) whose Axe is AX, supposing that the lateral surface of the Pyramid consists of Perimeters of triangles, parallel to the base BCD, these can neither be computed by the altitude AX, nor by the side AB, (for by the former, the thing required would be wanting of the true Dimension, and by the latter 'twould exceed it) but by the line AE drawn from the vertex A perpendicular to the side BC of the base: The reason of which is, that every plane side of a Pyramid, as ABC, consists of parallel right-lines computed by the altitude AE. After the same manner, supposing that the superficies of the Hemisphere BAD, (*Fig. 17.*) consists of Peripheries of circles parallel to the base BCD, the number of them is not to be computed by the Axis AX, but by the Quadrantal Arc AB, because that every point of the Arc AD in revolving produces a circumference; and so any superficies, whether plane or curv'd, which is conceived to consist of equidistant right or curv'd lines, is to be computed by a line cutting those equidistant lines perpendicularly. For since those equidistant lines, in this *Method of Indivisibles*, are not considered absolutely as lines having an infinitely small breadth, which is the same with the breadth or thickness of the point describing those equidistant lines in their Circumvolution, and since the same equidistant lines divide the line cutting them perpendicularly into parts measuring its breadth, those parts are to be looked upon as such sort of points, and consequently the number of equidistant



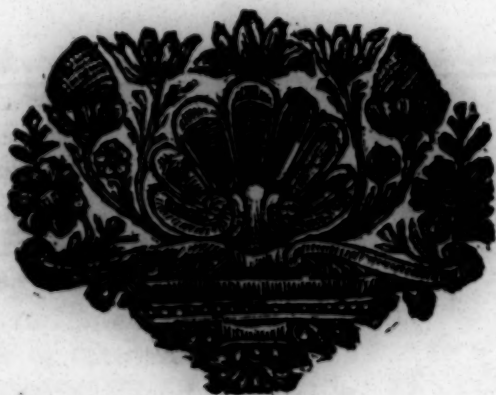
distant lines, or the sum of those breadths is to be computed by the number of points in the line cutting them perpendicularly, that is, by the length of that line, and not by a line of any other length, for that will consist of more or less points.

Hence therefore in the speculation of the superficies of solids, the *Method of Indivisibles* is not unuseful, but rather very commodious, provided it be rightly understood, and applied according to the Rule prescribed. For by the help of it even these superficies may be found, if we have some convenient *Data* presupposed, on which the reasoning may be founded: For instance, we might, by the help of it, investigate the superficies of a Cone, by reasoning after this manner.

If the superficies of the Cone ABC ( *Fig. 15.* ) be divided into innumerable Peripheries of circles  $b \times d$  parallel to the base BCD, the breadth of those Peripheries taken together, make up the side AB cutting them perpendicularly, and consequently there will be as many Peripheries as there are points in the line AB, that is, their number may be express'd by the number of points in AB, or by its length. Wherefore if you draw perpendiculars equal to the Peripheries to every point of AB, a superficies will be made out of those perpendiculars equal to the superficies of the Cone; but that superficies will be a triangle whose height is AB, and base equal to the greatest Periphery BDC, and so the superficies of the Cone will be  $= \frac{1}{2} AB \times \text{Periph. BDC}$ , which conclusion agrees with the things laid down and demonstrated by *Archimedes*.

After the same manner, if you take any right-line  $ab$  ( *Fig. 18.* ) equal to the quadrantal Arc AB of the Hemisphere ( *Fig. 17.* ) and to each of its points  $m$  let the perpendiculars  $mn$  be erected equal to the Radii MN of the parallel circles MOM passing through the corresponding points M of that quadrantal Arc, the greatest of which  $b \times$  let be equal to the Radius BX of the base of the Hemisphere: The figure  $abx$  will contain the Radii of all the circles of whose Peripheries the superficies of the Hemisphere consists. And if the perpendicular  $mo$ ,  $bd$  be erected equal to the Peripheries MO, BD, there will be made the figure  
 $abd$

$a b d$  equal to the superficies of the Hemisphere. The dimension of which figure if you can by any means find (as in this case you are to find the Area of the figure  $a b x$ ) thence you will easily deduce the content of the segment of the sphere, agreeing to what you would gather by any other genuine method of reasoning. Which Observation, I think, will not be unuseful in *Geometry*.



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# A N APPENDIX,

CONTAINING

The NATURE, CONSTRUCTION,  
and APPLICATION

O F

## LOGARITHMS.

**L**ogarithms are a set of artificial numbers so proportioned among themselves, and adapted to the natural numbers, so as to perform the same things by addition and subtraction, as these do by multiplication and division.

From this definition it follows, 1. That in any system or table of Logarithms whatever, the Logarithm of unity or 1, will be nothing: for as one neither increases nor diminishes the number multiplied by it, so neither will its Logarithm either increase or diminish the Logarithm to which it is added; and consequently the Logarithm of 1 must be nothing.

2. For a like reason, the Logarithm of a proper fraction will always be negative; for such a fraction always diminishes the number multiplied by it, and therefore its number will always diminish the Logarithm to which it is added.

3. This property of Logarithms, affords us no small compendium in multiplication; for whenever one number is to be multiplied by another, it is but taking out their Logarithms, and adding them together, and their sum will be a third Logarithm, whose natural number being taken out of the tables will be the product required.

4. The subtraction of Logarithms answers to the division of natural numbers to which they belong; that is, whenever one number is to be divided by another, it is but subtracting the logarithm of the divisor from the Logarithm of the dividend, and the remainder will be the Logarithm

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of

of the quotient; and because every fraction is nothing else but the quotient of the numerator divided by the denominator, its Logarithm will be found by subtracting the Logarithm of the denominator from the Logarithm of the numerator. To demonstrate this let the number  $x$  be divided by the number  $y$ ; and let the quotient be the number  $z$ ; and let the Logarithms of the numbers  $x, y, z$ , be  $a, b, c$ , respectively; I say then that  $a - b = c$ ; for since by the supposition  $\frac{x}{y} = z$ , we shall have  $x = yz$ , and  $a = b + c$  by the definition; whence  $a - b = c$ .

5. As every fourth proportional is found by multiplying the second and third numbers together, and dividing the product by the first, so the Logarithm of every such fourth proportional will be found by adding the Logarithms of the second and third numbers together, and subtracting from the sum the Logarithm of the first. This renders all the operations by the rule of proportion very compendious and easy; this compendium is chiefly useful in Trigonometry, both plain and spherical, which it renders very easy to be performed.

6. If  $x$  be any number, whose Logarithm is  $a$ , then the Logarithm  $x^2$  will be  $2a$ , that of  $x^3$ ,  $3a$ , &c. that of  $\frac{1}{x}$   $-a$ , that of  $\frac{1}{x^2}$   $-2a$ , &c. And universally, the Logarithm of  $x^m$  will be  $a \times m$ , and that, whether the index  $m$  be integral or fractional, affirmative or negative: on the other hand if  $q$  be the Logarithm of any power of  $x$ , as of  $x^m$ , then  $\frac{q}{m}$  will be the Logarithm of  $x$ . The reason of all this is plain; for as  $x^2$  is the product of  $x$  multiplied into itself, so its Logarithm will be the Logarithm of  $x$  added to itself or doubled, that is  $2a$ ; and so of the higher powers. Again, as  $\frac{1}{x}$  is the quotient of unity divided by  $x$ , its Logarithm will be found by subtracting  $a$  from  $0$ , the Logarithm of  $1$ , which gives  $-a$ ; and so of the other powers. Lastly, as  $\sqrt{x}$ , when multiplied into itself produces  $x$ , so its Logarithm when added to itself ought to make  $a$ ; therefore the Logarithm of  $\sqrt{x}$  will be  $\frac{1}{2}a$ , and so of all the other fractional powers.

7. If any set of numbers  $1, x, xx, x^3$ , &c. be in continual geometrical progression, their Logarithms,  $0, a, b, c$ , &c.



&c. will be in arithmetical progression: For since by the supposition 1 is to  $x$ , as  $x$  is to  $xx$ , as  $xx$  is to  $x^3$ ; that is, since  $\frac{x}{1} = \frac{xx}{x} = \frac{x^3}{xx}$ , we shall have  $a - o = b - a = c - b$  by the fourth consectory; therefore  $o$ ,  $a$ ,  $b$ ,  $c$ , are in arithmetical progression.

In the progression series of geometrical proportionals, if between the terms 1 and  $x$  be inserted a mean proportional, which is  $\sqrt{x}$ , its index or Logarithm will be  $\frac{1}{2}$ , because its distance from unity will be but one half of the distance of  $x$  from unity, and consequently the root of  $x$  will be expressed by  $\frac{1}{2}$ . In like manner, if between  $x$  and  $x^2$  we insert a mean proportional, its index will be 1 and  $\frac{1}{2}$  or  $\frac{3}{2}$ , because its distance from unity is once and half the distance of  $x$  from the same place of unity.

Again, if between 1 and  $x$  be inserted two mean proportionals; the first of these will be the cube root of  $x$  or  $\sqrt[3]{x}$ , and its index will be  $\frac{1}{3}$ , because its distance from the units place is but one third of the distance of  $x$  from the same place of unity; and consequently  $\sqrt[3]{x}$  is expressed by  $x^{\frac{1}{3}}$ ; whence it follows, that the index, or Logarithm of unity is nothing, as we have before observed; because unity cannot be removed at any distance from itself.

And the same series of geometrical proportionals may be continued on the contrary side of unity, or towards the left-hand, and which will therefore decrease in the same proportion as the terms placed on the right-hand increases.

For the terms  $\frac{1}{x^3}$ ,  $\frac{1}{x^2}$ ,  $\frac{1}{x}$ ,  $\frac{1}{x^2}$ ,  $\frac{1}{x^3}$ , 1,  $x$ ,  $x^2$ ,  $x^3$ ,  $x^4$ ,  $x^5$ , &c. are in the same geometrical progression continued, and have all the same common ratio; therefore because the distance of  $x$  from unity, towards the right-hand is positive, or  $+1$ , the distance of  $\frac{1}{x}$  from unity towards the left-hand, which is equal to the former, is negative, or  $-1$ , consequently the index of  $\frac{1}{x}$  may be written  $x^{-1}$ .

Also, because  $x^2$  is on the right side of unity, and its index  $+2$ , the index of  $\frac{1}{x^2}$ , which is as far distant from unity on the contrary side, or towards the left-hand, will be  $-2$ , and consequently  $\frac{1}{x^2}$  is the same with  $x^{-2}$ . In



like manner  $x^{-3}$  is the same with  $\frac{1}{x^3}$ , and  $x^{-4}$  the same with  $\frac{1}{x^4}$ ; whence it follows that these negative indices shew that the terms to which they belong, are as far distant from unity towards the left-hand, as the terms whose indices are the same and positive, are removed from unity on the contrary side, or towards the right-hand.

Again, if between 1 and  $x$ , in the series increasing from unity there be inserted a mean proportional as  $\sqrt{x}$ , and between this mean proportional, and unity be inserted another mean proportional, as  $\sqrt[4]{x}$ , and between this last mean proportional and unity be inserted another mean proportional as  $\sqrt[8]{x}$ , &c. those respective indices will be  $\frac{1}{2}$ ,  $\frac{1}{4}$ ,  $\frac{1}{8}$ , &c. likewise if between unity and the next term  $\frac{1}{x}$  decreasing be inserted a mean proportional, as  $\frac{1}{\sqrt{x}}$ , the correspondent index will be  $-\frac{1}{2}$ ; and if between this mean proportional and unity be inserted another mean proportional

as  $\frac{1}{\sqrt[4]{x}}$  the correspondent index will be  $-\frac{1}{4}$ , &c. And

as the same may be done between any other two terms of the progression, it follows, that between any two terms may be inserted an infinite number of mean proportionals, whose respective indices will become the Logarithms of the respective terms to which they belong: And these mean proportionals the greater they are in number, between the two terms of any ratio, the nearer they approach to a ratio of equality, with their correspondent indices or exponents.

Hence we see the reason of the ancient method of making Logarithms, and which was practiced by their first inventors, which was, to extract the square root, out of the square root, &c. of any number, in order to find a series of continual mean geometrical proportionals; till the number of cyphers contained between unity, and the first significant figure of the root, was equal to the number of places the intended Logarithm should consist of; and at the same time to find out a like number of correspondent arithmetical means, which would be the Logarithms of the geometrical means respectively.

But how tedious and opperose this method is, may be easily imagined, by any one who has been at the pains to extract the root of a large number; and may be gathered from this, that to find a Logarithm to seven places only, requires

requires at least 27 extractions of root out of root, the square consisting of 16 figures at least; and if in any one of the operations an error happens, the whole work must be repeated.

From what has been said we may observe that the Logarithm of any number, is the Logarithm of the ratio of unity, to the number itself; or it is the distance between unity and the same number, in the geometrical scale of proportionals, and is measured by the number of proportionals contained between them: Logarithms therefore expound the place or order that every number obtains from the units place in the continued series or scale of proportionals, infinite in number.

Thus, if between unity and the number 10, there be supposed 10,000,000, &c. mean proportionals, that is, if the number 10 be placed in the 10,000,000th place from unity, between 1 and 2 there will be found 3010300 of such proportionals; that is, the number 2 will be placed in the 3010300th place from unity; between 1 and 3, there will be found 4771213 of the same proportionals, or the number 3 will stand in the 4771213th place, in the infinite scale of proportionals, which numbers 10000000, 3010300, 4771213, are therefore the Logarithms of 10, 2, and 3; or, more properly, the Logarithms of the ratio's of those numbers one to the other.

Again, if between unity and the number 10 there be supposed an infinite scale of mean proportionals, whose number is 23025851, &c. that is, if the number 10 be placed in the 23025851st place from unity, in the infinite scale of proportionals between 1 and 2 there will be found to be 6931471, &c. such proportionals; and between 1 and three 10986122, &c. of the same proportionals; so that if the first term of the series be called  $x$ , the second term will be  $x^2$ , the third  $x^3$ , &c. and if the number 10 be supposed to be the 10000000th term in the series, the number 2 will be the 3010300th term, and the number 3 will be the 4771213th term in the same series; and consequently  $x^{10000000} = 10$ ,  $x^{3010300} = 2$ , and  $x^{4771213} = 3$ , &c. and again, if the number 10 be supposed the 23025851st term, in the series, the number 2 will be the 6931471st term, and the number 3, the 10986122nd term in the same series; and consequently, in this case,  $x^{23025851} = 10$ ,  $x^{6931471} = 2$ , and  $x^{10986122} = 3$ , &c.

And as the infinite number of mean proportionals between any two numbers may be assumed at pleasure, hence Logarithms may be of as many different forms (that is, there may be as many different scales of Logarithms) as there



can be assumed different scales of mean proportionals between any two numbers. Every number therefore is some certain power of that number which is placed next to unity, in the infinite scale of proportionals between unity and the given number, and the index of that power is the Logarithm of the number.

Logarithms therefore may be of as many sorts as you can assume different indices of the power of that number whose Logarithm is required,

And because the ratio of unity to any number is measured by the number of ratiuncula contained in that ratio; we may therefore value ratio's by the number of ratiuncula contained in each ratio, and may consider them as *Quantates sui generis*, beginning from the ratio of 1 to 1 = 0, being affirmative when the root is increasing, but negative when decreasing,

So that ratio's are to each other as the number of like and equal ratiuncula contained between the two terms of the ratio; wherefore, if the ratio of 1 to 10 contain 10000000, &c. equal ratiuncula, that of its duplicate, or of 10 to 100, will contain 20000000, &c. that of its triplicate, or of 100 to 1000, 30000000, &c. of the same similar and equal ratiuncula; so that the Logarithms of numbers, or the values of ratio's, are in an arithmetical progression; and hence ariseth the common definition of Logarithms, viz. that Logarithms are a series of numbers in an arithmetical progression fitted or assigned to a rank of numbers in a geometrical progression.

If the distance between unity and any number be supposed to consist always of the same number of mean proportionals as 10000000, &c. or, which is the same thing, if we make the ratio of unity to any number, to consist always of the same infinite number of ratiuncula, the magnitudes of each in this case, will be, as their number in the former; so that if between unity and any number proposed, there be taken an infinity of mean proportionals, the infinitely small augment or decrement of the first of those means from unity, will be a ratiuncula; and because this is always proportional to the momentum or fluxion of the ratio of unity to the said number; and since in these continual proportionals all the ratiuncula are equal, their sum, or the whole ratio, will be as the said momentum or fluxion is directly; wherefore if the root of the infinite powers be extracted out of any number, the difference between that root and unity, will be as the Logarithm of that number; so that the infinite root of any number being found, the Logarithm of that number is



is easily computed. How the infinite root of any number may be found, must be next considered.

Let  $p + px$  be a given quantity whose  $n$  root is to be extracted; where  $n$  stands for any power or number taken at pleasure.

$$\text{Then will } p + px = p \times 1 + x.$$

$$\text{Assume } 1 + x = 1 + Ax + Ba^2 + Cx^3 + Dx^4 + Ex^5, \text{ \&c.}$$

$$\text{Then } n \times 1 + x = Ax + 2Bx^2 + 3Cx^3 + 4Dx^4 + 5Ex^5, \text{ \&c.}$$

$$\text{And } n \times 1 + x = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4, \text{ \&c.}$$

$$n \times 1 + x = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4, \text{ \&c.}$$

$$\text{And } \frac{1 + x}{1 + x} = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5, \text{ \&c.}}{1 + x}$$

$$\text{And } n + nAx + nBx^2 + nCx^3 + nDx^4, \text{ \&c.} = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4, \text{ \&c.}$$

$$\text{Also } n = A, \text{ and } nA = 2B + A. \text{ Also } nB = 3C + 2B.$$

$$\text{And } nA - A = 2B. \text{ And } nB - 2B = 3C.$$

$$\text{And } \frac{n-1}{1} \times A = 2B. \text{ And } \frac{n-2}{2} \times B = 3C.$$

$$\text{And } \frac{n-1}{2} \times A = B. \text{ And } \frac{n-2}{3} \times B = C.$$

$$\text{Also } nC = 4D + 3C,$$

$$\text{Also } nD = 5E + 4D.$$

$$\text{And } nC - 3C = 4D,$$

$$\text{And } nD - 4D = 5E.$$

$$\text{And } \frac{n-3}{1} \times C = 4D,$$

$$\text{And } \frac{n-4}{1} \times D = 5E.$$

$$\text{And } \frac{n-3}{4} \times C = D.$$

$$\text{And } \frac{n-4}{5} \times D = E.$$

$$\text{Wherefore } A = n, B = n \times \frac{n-1}{2}, C = n \times \frac{n-1}{2} \times \frac{n-2}{3},$$

$$D = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}, \text{ and } E = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}, \text{ \&c.}$$

$$\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}, \text{ \&c.}$$

Z 4

And

$$\text{And, } 1+x = 1 + nx + n \times \frac{n-1}{2} x^2 + n \times \frac{n-2}{2} x^3 + n \times \frac{n-3}{4} x^4, \text{ \&c.}$$

$$\text{And } p+px = p^n + n p^{n-1} x + n \times \frac{n-1}{2} p^{n-2} x^2 + n \times \frac{n-2}{3} p^{n-3} x^3, \text{ \&c.}$$

$$\text{Put } p^n = A, np^{n-1} = B, n \times \frac{n-1}{2} p^{n-2} = C, n \times \frac{n-2}{3} p^{n-3} = C.$$

$$\text{Then } p+px = p^n + nAx + \frac{n-1}{2} Bx + \frac{n-2}{3} Cx + \frac{n-4}{5} Ex, \text{ \&c.}$$

Which is the celebrated binomial series invented by the illustrious Sir *Isaac Newton*, for raising a given quantity to any given power, or extracting any root out of the same quantity; because  $n$  stands for any number whether it be a whole number or a fraction. When

$$n \text{ is finite, } 1+x^n = 1 + nx + \frac{1-n}{2n} x^2 + \frac{1-3n+2n^2}{6n^3} x^3 + \frac{1-6n+11n^2-6n^3}{24n^4} x^4, \text{ \&c.}$$

$$\text{but when it is infinite } 1+x^n = 1 + \frac{1}{n}x - \frac{1}{2n^2}x^2 + \frac{1}{3n^3}x^3 - \frac{1}{4n^4}x^4 + \frac{1}{5n^5}x^5, \text{ \&c.}$$

$nn$  being infinitely infinite, and consequently whatever is divided by it vanishes; whence it follows, that  $\frac{1}{n}$  multiplied into  $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4, \text{ \&c.}$  is the augment of the first of the mean proportionals between unity and  $1+x$ , and is therefore the Logarithm of the ratio of 1 to  $1+x$ .

$$\text{But because the infinite root of } 1-x, \text{ or } 1-x \text{ is } 1 - \frac{1}{n}x + \frac{1}{2n^2}x^2 - \frac{1}{3n^3}x^3 + \frac{1}{4n^4}x^4, \text{ \&c. there-}$$

fore

fore  $\frac{1}{n} \times$  into  $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5$ , &c. will be the Logarithm of the ratio of 1 to  $1-x$ , or the Logarithm of a number less than unity; and whereas the infinite index  $n$  may be taken at pleasure, the several scales of Logarithms to such indices will be reciprocally as  $\frac{1}{n}$ , or as the several indices, and if the index be taken  $= 10000000$ , &c. the respective Logarithms will be simply,

$$x + \frac{1}{2n}x^2 + \frac{1}{3n}x^3 + \frac{1}{4n}x^4 + \frac{1}{5n}x^5, \text{ \&c.}$$

Let  $a$  represent the least of any two given numbers,  $b$  the greatest,  $z$  the sum of the two numbers, and  $d$  the difference, and let us suppose the ratio of  $a$  to  $b$ , to be divided into that of  $a$  to  $\frac{1}{2}z$ , and of  $\frac{1}{2}z$  to  $b$ ; that is into the ratio of  $a$  to the arithmetical mean between the two numbers, and the ratio of the said arithmetical mean to the greater number  $b$ ; then because  $\frac{a}{\frac{1}{2}z} \times \frac{\frac{1}{2}z}{b} = \frac{a}{b}$ , the

Logarithm of  $\frac{a}{\frac{1}{2}z} + \text{Log. } \frac{\frac{1}{2}z}{b} = \text{Log. of } \frac{a}{b}$ , or because  $\frac{\frac{1}{2}z}{a} \times \frac{b}{\frac{1}{2}z} = \frac{b}{a}$  the Log. of  $\frac{\frac{1}{2}z}{a} + \text{Log. } \frac{b}{\frac{1}{2}z} = \text{Log. } \frac{b}{a}$ ; that is, the sum of the Logarithms of these two ratio's will be the Log. of the ratio of  $a$  to  $b$ ; and to find the value of each of these ratio's in the terms of 1 and  $1+x$  we must say, as  $\frac{1}{2}z$ :

$a :: 1; 1-x$ , whence  $x = \frac{b - \frac{1}{2}z}{\frac{1}{2}z} = \frac{d}{z}$ , and substituting  $\frac{d}{z}$

in the room of  $x$ , in each of the former series, we shall have  $\frac{1}{n} \times \frac{d}{z} + \frac{d^2}{2z^2} + \frac{d^3}{3z^3} + \frac{d^4}{4z^4} + \frac{d^5}{5z^5}$ , &c. for the Log.

of the ratio of  $a$  to  $\frac{1}{2}z$ , and  $\frac{1}{n} \times \frac{d}{z} - \frac{d^2}{2z^2} + \frac{d^3}{3z^3} - \frac{d^4}{4z^4} + \frac{d^5}{5z^5}$ , &c. for the Log. of the ratio  $\frac{1}{2}z$  to  $b$ ; and

consequently the sum of these two series  $\frac{1}{n} \times \frac{2d}{z} + \frac{2d^3}{3z^3} + \frac{2d^5}{5z^5}$ , &c. or  $\frac{1}{n} \times z \times \frac{d}{z} + \frac{d^3}{3z^3} + \frac{d^5}{5z^5} + \frac{d^7}{7z^7} + \frac{d^9}{9z^9}$ , &c. will be the Logarithm of the ratio of  $a$  to  $b$ , whose difference is  $d$ , and sum  $z$ ; whence to find the Logarithm of any prime number we have this general rule. To



" To the given number add 1 for a denominator or  
 " divisor, and subtract 1 from the same number for a  
 " numerator or dividend, then of the vulgar fraction thence  
 " resulting, compose all the odd powers thereof, these will  
 " form a series of numbers which being divided by their  
 " respective exponents, *viz.* 1, 3, 5, 7, 9, &c. will produce  
 " a series of quotes, whose sum will be the natural Loga-  
 " rithm of the number proposed, and being doubled, will  
 " give the Logarithm of the same number, according to  
 " *Neper's* form. "

This rule is not only very easy to be retained in memory, but very proper for the practice of making Logarithms, which it performs with the greatest expedition, and is so very plain that it may be taught to persons of a mean capacity.

For example, Let it be required to find the Logarithm of 2, the first prime number.

Because  $a = 1$ , and  $b = 2$ , therefore  $d = b - a = 1$ ,  
 and  $z = a + b = 3$ ; wherefore  $\frac{d}{z} = \frac{1}{3}$ , and  $\frac{d^2}{z^2} = \frac{1}{9}$ .

Hence  $\frac{1}{3} + \frac{1}{3} \times \frac{1}{27} + \frac{1}{5} \times \frac{1}{243} + \frac{1}{7} \times \frac{1}{2187}$ , &c.  
 $= \frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \frac{1}{15309}$ , will be the natural Loga-  
 rithm of 2, and consequently  $\frac{2}{3} + \frac{2}{81} + \frac{2}{1215} + \frac{2}{15309}$ ,

&c. will be the Logarithm of the same number in *Neper's* form; wherefore if the several fractions be added together, and reduced into an equivalent decimal, we shall have the Logarithm of the given number; but because to multiply by  $\frac{1}{9}$  and divide by 9 is the same thing, if the first fraction be reduced into a decimal, and that divided by 9, and each successive quotient by the same, we shall have a series of quotients, which being divided by the several co-efficients 1, 3, 5, 7, 9, &c. the sum of these last quotients will be the Logarithm of the number given; and in order to obtain the Logarithm true to any number of places, it will be necessary to continue the fraction to one or two more places, than the intended number of places the Logarithm is to consist of; so that to have the Logarithm true to 10 places, it will be convenient to find the value of  $\frac{1}{9}$  in decimals to 11 or 12 places, as in the following operation.

of any prime number we have the Logarithm of the same number

|                                  |                                  |                                  |                                  |
|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\frac{1}{3} = 333.333.333.333$  | $\frac{1}{3} = 333.333.333.333$  | $\frac{1}{3} = 333.333.333.333$  | $\frac{1}{3} = 333.333.333.333$  |
| $\frac{1}{4} = 250.000.000.000$  | $\frac{1}{4} = 250.000.000.000$  | $\frac{1}{4} = 250.000.000.000$  | $\frac{1}{4} = 250.000.000.000$  |
| $\frac{1}{5} = 200.000.000.000$  | $\frac{1}{5} = 200.000.000.000$  | $\frac{1}{5} = 200.000.000.000$  | $\frac{1}{5} = 200.000.000.000$  |
| $\frac{1}{6} = 166.666.666.666$  | $\frac{1}{6} = 166.666.666.666$  | $\frac{1}{6} = 166.666.666.666$  | $\frac{1}{6} = 166.666.666.666$  |
| $\frac{1}{7} = 142.857.142.857$  | $\frac{1}{7} = 142.857.142.857$  | $\frac{1}{7} = 142.857.142.857$  | $\frac{1}{7} = 142.857.142.857$  |
| $\frac{1}{8} = 125.000.000.000$  | $\frac{1}{8} = 125.000.000.000$  | $\frac{1}{8} = 125.000.000.000$  | $\frac{1}{8} = 125.000.000.000$  |
| $\frac{1}{9} = 111.111.111.111$  | $\frac{1}{9} = 111.111.111.111$  | $\frac{1}{9} = 111.111.111.111$  | $\frac{1}{9} = 111.111.111.111$  |
| $\frac{1}{10} = 100.000.000.000$ | $\frac{1}{10} = 100.000.000.000$ | $\frac{1}{10} = 100.000.000.000$ | $\frac{1}{10} = 100.000.000.000$ |
| $\frac{1}{11} = 90.909.090.909$  | $\frac{1}{11} = 90.909.090.909$  | $\frac{1}{11} = 90.909.090.909$  | $\frac{1}{11} = 90.909.090.909$  |
| $\frac{1}{12} = 83.333.333.333$  | $\frac{1}{12} = 83.333.333.333$  | $\frac{1}{12} = 83.333.333.333$  | $\frac{1}{12} = 83.333.333.333$  |
| $\frac{1}{13} = 76.923.076.923$  | $\frac{1}{13} = 76.923.076.923$  | $\frac{1}{13} = 76.923.076.923$  | $\frac{1}{13} = 76.923.076.923$  |
| $\frac{1}{14} = 71.428.571.428$  | $\frac{1}{14} = 71.428.571.428$  | $\frac{1}{14} = 71.428.571.428$  | $\frac{1}{14} = 71.428.571.428$  |
| $\frac{1}{15} = 66.666.666.666$  | $\frac{1}{15} = 66.666.666.666$  | $\frac{1}{15} = 66.666.666.666$  | $\frac{1}{15} = 66.666.666.666$  |
| $\frac{1}{16} = 62.500.000.000$  | $\frac{1}{16} = 62.500.000.000$  | $\frac{1}{16} = 62.500.000.000$  | $\frac{1}{16} = 62.500.000.000$  |
| $\frac{1}{17} = 58.823.529.411$  | $\frac{1}{17} = 58.823.529.411$  | $\frac{1}{17} = 58.823.529.411$  | $\frac{1}{17} = 58.823.529.411$  |
| $\frac{1}{18} = 55.555.555.555$  | $\frac{1}{18} = 55.555.555.555$  | $\frac{1}{18} = 55.555.555.555$  | $\frac{1}{18} = 55.555.555.555$  |
| $\frac{1}{19} = 52.631.578.947$  | $\frac{1}{19} = 52.631.578.947$  | $\frac{1}{19} = 52.631.578.947$  | $\frac{1}{19} = 52.631.578.947$  |
| $\frac{1}{20} = 50.000.000.000$  | $\frac{1}{20} = 50.000.000.000$  | $\frac{1}{20} = 50.000.000.000$  | $\frac{1}{20} = 50.000.000.000$  |
| $\frac{1}{21} = 47.619.047.619$  | $\frac{1}{21} = 47.619.047.619$  | $\frac{1}{21} = 47.619.047.619$  | $\frac{1}{21} = 47.619.047.619$  |
| $\frac{1}{22} = 45.454.545.454$  | $\frac{1}{22} = 45.454.545.454$  | $\frac{1}{22} = 45.454.545.454$  | $\frac{1}{22} = 45.454.545.454$  |
| $\frac{1}{23} = 43.478.260.869$  | $\frac{1}{23} = 43.478.260.869$  | $\frac{1}{23} = 43.478.260.869$  | $\frac{1}{23} = 43.478.260.869$  |
| $\frac{1}{24} = 41.666.666.666$  | $\frac{1}{24} = 41.666.666.666$  | $\frac{1}{24} = 41.666.666.666$  | $\frac{1}{24} = 41.666.666.666$  |
| $\frac{1}{25} = 40.000.000.000$  | $\frac{1}{25} = 40.000.000.000$  | $\frac{1}{25} = 40.000.000.000$  | $\frac{1}{25} = 40.000.000.000$  |
| $\frac{1}{26} = 38.461.538.461$  | $\frac{1}{26} = 38.461.538.461$  | $\frac{1}{26} = 38.461.538.461$  | $\frac{1}{26} = 38.461.538.461$  |
| $\frac{1}{27} = 37.037.037.037$  | $\frac{1}{27} = 37.037.037.037$  | $\frac{1}{27} = 37.037.037.037$  | $\frac{1}{27} = 37.037.037.037$  |
| $\frac{1}{28} = 35.714.285.714$  | $\frac{1}{28} = 35.714.285.714$  | $\frac{1}{28} = 35.714.285.714$  | $\frac{1}{28} = 35.714.285.714$  |
| $\frac{1}{29} = 34.482.758.620$  | $\frac{1}{29} = 34.482.758.620$  | $\frac{1}{29} = 34.482.758.620$  | $\frac{1}{29} = 34.482.758.620$  |
| $\frac{1}{30} = 33.333.333.333$  | $\frac{1}{30} = 33.333.333.333$  | $\frac{1}{30} = 33.333.333.333$  | $\frac{1}{30} = 33.333.333.333$  |

Whence the natural Log. of 2 is 346.573.590.280

And the Neper's Log. of 2 is 693.147.180.560

Hence if the Neper's Log. of 2 be doubled will give 1386.294.361.120, for the Neper's Log. of 4, and this again being doubled will give 2772.588.722.240, for the Neper's Log. of 16, &c. Or if the Log. of 2 be trebled, or multiplied by 3, we shall have 2079.441.541.680, for the Log. of 8, &c.

And, if the Logarithm of 8 be multiplied by the Log. of  $1\frac{1}{4}$ , we shall have the Log. of 10; and to find the Log.  $1\frac{1}{4}$  we have  $a=1$ ,  $b=1\frac{1}{4}=\frac{5}{4}$ , whence  $x=\frac{2}{4}$ , and  $d=\frac{1}{4}$ , and consequently  $\frac{d}{x}=\frac{1}{2}$ , and  $\frac{d^2}{x^2}=\frac{1}{8}$ . From whence the Log. of 10 will be found as follows.

|                                  |                                  |                                  |                                  |
|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\frac{1}{3} = 333.333.333.333$  | $\frac{1}{3} = 333.333.333.333$  | $\frac{1}{3} = 333.333.333.333$  | $\frac{1}{3} = 333.333.333.333$  |
| $\frac{1}{4} = 250.000.000.000$  | $\frac{1}{4} = 250.000.000.000$  | $\frac{1}{4} = 250.000.000.000$  | $\frac{1}{4} = 250.000.000.000$  |
| $\frac{1}{5} = 200.000.000.000$  | $\frac{1}{5} = 200.000.000.000$  | $\frac{1}{5} = 200.000.000.000$  | $\frac{1}{5} = 200.000.000.000$  |
| $\frac{1}{6} = 166.666.666.666$  | $\frac{1}{6} = 166.666.666.666$  | $\frac{1}{6} = 166.666.666.666$  | $\frac{1}{6} = 166.666.666.666$  |
| $\frac{1}{7} = 142.857.142.857$  | $\frac{1}{7} = 142.857.142.857$  | $\frac{1}{7} = 142.857.142.857$  | $\frac{1}{7} = 142.857.142.857$  |
| $\frac{1}{8} = 125.000.000.000$  | $\frac{1}{8} = 125.000.000.000$  | $\frac{1}{8} = 125.000.000.000$  | $\frac{1}{8} = 125.000.000.000$  |
| $\frac{1}{9} = 111.111.111.111$  | $\frac{1}{9} = 111.111.111.111$  | $\frac{1}{9} = 111.111.111.111$  | $\frac{1}{9} = 111.111.111.111$  |
| $\frac{1}{10} = 100.000.000.000$ | $\frac{1}{10} = 100.000.000.000$ | $\frac{1}{10} = 100.000.000.000$ | $\frac{1}{10} = 100.000.000.000$ |
| $\frac{1}{11} = 90.909.090.909$  | $\frac{1}{11} = 90.909.090.909$  | $\frac{1}{11} = 90.909.090.909$  | $\frac{1}{11} = 90.909.090.909$  |
| $\frac{1}{12} = 83.333.333.333$  | $\frac{1}{12} = 83.333.333.333$  | $\frac{1}{12} = 83.333.333.333$  | $\frac{1}{12} = 83.333.333.333$  |
| $\frac{1}{13} = 76.923.076.923$  | $\frac{1}{13} = 76.923.076.923$  | $\frac{1}{13} = 76.923.076.923$  | $\frac{1}{13} = 76.923.076.923$  |
| $\frac{1}{14} = 71.428.571.428$  | $\frac{1}{14} = 71.428.571.428$  | $\frac{1}{14} = 71.428.571.428$  | $\frac{1}{14} = 71.428.571.428$  |
| $\frac{1}{15} = 66.666.666.666$  | $\frac{1}{15} = 66.666.666.666$  | $\frac{1}{15} = 66.666.666.666$  | $\frac{1}{15} = 66.666.666.666$  |
| $\frac{1}{16} = 62.500.000.000$  | $\frac{1}{16} = 62.500.000.000$  | $\frac{1}{16} = 62.500.000.000$  | $\frac{1}{16} = 62.500.000.000$  |
| $\frac{1}{17} = 58.823.529.411$  | $\frac{1}{17} = 58.823.529.411$  | $\frac{1}{17} = 58.823.529.411$  | $\frac{1}{17} = 58.823.529.411$  |
| $\frac{1}{18} = 55.555.555.555$  | $\frac{1}{18} = 55.555.555.555$  | $\frac{1}{18} = 55.555.555.555$  | $\frac{1}{18} = 55.555.555.555$  |
| $\frac{1}{19} = 52.631.578.947$  | $\frac{1}{19} = 52.631.578.947$  | $\frac{1}{19} = 52.631.578.947$  | $\frac{1}{19} = 52.631.578.947$  |
| $\frac{1}{20} = 50.000.000.000$  | $\frac{1}{20} = 50.000.000.000$  | $\frac{1}{20} = 50.000.000.000$  | $\frac{1}{20} = 50.000.000.000$  |
| $\frac{1}{21} = 47.619.047.619$  | $\frac{1}{21} = 47.619.047.619$  | $\frac{1}{21} = 47.619.047.619$  | $\frac{1}{21} = 47.619.047.619$  |
| $\frac{1}{22} = 45.454.545.454$  | $\frac{1}{22} = 45.454.545.454$  | $\frac{1}{22} = 45.454.545.454$  | $\frac{1}{22} = 45.454.545.454$  |
| $\frac{1}{23} = 43.478.260.869$  | $\frac{1}{23} = 43.478.260.869$  | $\frac{1}{23} = 43.478.260.869$  | $\frac{1}{23} = 43.478.260.869$  |
| $\frac{1}{24} = 41.666.666.666$  | $\frac{1}{24} = 41.666.666.666$  | $\frac{1}{24} = 41.666.666.666$  | $\frac{1}{24} = 41.666.666.666$  |
| $\frac{1}{25} = 40.000.000.000$  | $\frac{1}{25} = 40.000.000.000$  | $\frac{1}{25} = 40.000.000.000$  | $\frac{1}{25} = 40.000.000.000$  |
| $\frac{1}{26} = 38.461.538.461$  | $\frac{1}{26} = 38.461.538.461$  | $\frac{1}{26} = 38.461.538.461$  | $\frac{1}{26} = 38.461.538.461$  |
| $\frac{1}{27} = 37.037.037.037$  | $\frac{1}{27} = 37.037.037.037$  | $\frac{1}{27} = 37.037.037.037$  | $\frac{1}{27} = 37.037.037.037$  |
| $\frac{1}{28} = 35.714.285.714$  | $\frac{1}{28} = 35.714.285.714$  | $\frac{1}{28} = 35.714.285.714$  | $\frac{1}{28} = 35.714.285.714$  |
| $\frac{1}{29} = 34.482.758.620$  | $\frac{1}{29} = 34.482.758.620$  | $\frac{1}{29} = 34.482.758.620$  | $\frac{1}{29} = 34.482.758.620$  |
| $\frac{1}{30} = 33.333.333.333$  | $\frac{1}{30} = 33.333.333.333$  | $\frac{1}{30} = 33.333.333.333$  | $\frac{1}{30} = 33.333.333.333$  |

The natural Log. of  $1\frac{1}{4}$  is 111.571.775.657

And the Neper's Log. of  $1\frac{1}{4}$  is 223.143.551.314

The Log. last found being added to the Log. of 8 found before, we shall have 2302.585.692.994, for the Neper's Log. of 10.

But as neither the natural Logarithms, nor those of Neper, the celebrated inventor of Logarithms, are so proper for calculation as could be wished, the Lord Neper himself, with the assistance of Mr. Henry Briggs, altered their form, and made the Logarithm of 10 to be 1,0000000000, &c. and



and not 2.30258509299, &c. that is, he made the number 10 the 10000000000th term in the series, whence the Logarithm of 100 will be 20000000000, &c. and the Logarithm of 1000, will be 30000000000, &c. whence the Logarithms between 1 and 10 will begin from 0, that is, they will have 0 for the first term towards the left-hand, because they are each of them less than the Logarithm of 10, which has unity or 1 in the first place; in like manner the Logarithms of all numbers betwixt 100 and 1000 begin from 2, because they are greater than the Logarithm of 100, which is fixed at 2, and less than the Logarithm of 1000, which is made equal to 3, &c.

The first figure of every Logarithm towards the left-hand, which is separated from the rest by a point, is called the index of that Logarithm; because it points out the highest or remotest place of that number from the place of unity in the infinite scale of proportionals towards the left-hand. Thus if the index of the Logarithm be one, it shews that its highest place towards the left-hand is the tenth place from unity; and therefore all Logarithms which have 1 for their index will be found between the tenth and hundredth place in the order of numbers. And for the same reason all Logarithms which have 2 for their index will be found between the hundredth and thousandth place in the order of numbers, &c. whence the index or characteristic of any Logarithm is always less by one than the number of figures in whole numbers which answer to the given Logarithm.

As all systems of Logarithms whatever are composed of similar quantities, it will be easy to form, from any system of Logarithms another system in any given ratio, and consequently to reduce one table of Logarithms into another of any given form. As for instance, in the system given let  $a$ ,  $b$ , and  $c$  be the Logarithms of the three given numbers  $A$ ,  $B$ , and  $C$ , whereof the third is equal to the product of the other two multiplied together. Then will  $a + b = c$  by the definition we have given of Logarithms. Let us now imagine all the Logarithms of this given system to be doubled, then will  $a$ ,  $b$ , and  $c$  be changed into  $2a$ ,  $2b$ ,  $2c$ ; but as  $a + b$  was equal to  $c$  in the former system, so in the latter  $2a + 2b$  will be equal to  $2c$ . Whence it follows that we may reduce one system of Logarithms into another of any given form by saying, as any one Logarithm in the given form is to its correspondent Logarithm in the new form; so is any other Logarithm in the given form, to its correspondent Logarithm in the required form; and hence we are taught how to reduce either *Neper's* or the natural Logarithms into the form of *Brigg's*, and the contrary. For



For as 2. 302. 585. 092. 994, the *Neper's* Logarithm of 10, to 1.0000000000, the *Brigg's* Logarithm of 10; so is any other Logarithm in *Neper's* form, to the correspondent tabular Logarithm in *Brigg's* form: And because the two first numbers constantly remain the same; if the *Neper's* Logarithm of any one number be divided by 2.302.585, &c. or multiplied by 434. 294. 481. 903, the ratio of 1.00000, &c. to 2. 302. 58, &c. as is found by dividing 1.000000, &c. by 2. 302. 58, &c. the quotient in the former, and the product in the latter will give the correspondent Logarithm in *Brigg's* form, and the contrary. And after the same manner the ratio of natural Logarithms to that of *Brigg's* will be found = 868. 588. 963. 806.

If instead of finding a decimal fraction equal to  $\frac{d}{z}$  we divide the reciprocal ratio of the *Neper's* or the natural Logarithms, to that of *Brigg's* by the reciprocal ratio of  $d$  to  $z$ , and divide that quotient by the square of  $z$ , we shall have a series of quotients, which being divided respectively by the several co-efficients 1, 3, 5, 7, 9, &c. will produce a new series of quotients, whose sum will give the *Brigg's* Logarithm of the number proposed.

Let it therefore be required to find the *Brigg's* Logarithm of 2 to ten places, from the reciprocal ratio of the natural to the *Brigg's* Logarithm.

Put  $R=868. 588. 963. 806$ , and because  $a=1, b=2, d=1$ , and  $z=3$ ; therefore  $R \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{27} + \frac{1}{5} \times \frac{1}{243} + \frac{1}{7} \times \frac{1}{2187}$  &c. = the Logarithm of 2.

|                                                     |                       |
|-----------------------------------------------------|-----------------------|
| $\frac{1}{3} R = 289. 529. 654. 60$                 | $= 289. 529. 654. 60$ |
| $\frac{1}{27} = 32. 169. 961. 62$ its $\frac{1}{3}$ | $= 10. 723. 320. 54$  |
| $\frac{1}{243} = 3. 574. 440. 18$ its $\frac{1}{3}$ | $= 714. 888. 04$      |
| $\frac{1}{2187} = 397. 160. 02$ its $\frac{1}{3}$   | $= 56. 737. 14$       |
| $\frac{1}{19683} = 44. 128. 89$ its $\frac{1}{3}$   | $= 4. 903. 21$        |
| $\frac{1}{177147} = 4. 903. 21$ its $\frac{1}{3}$   | $= 445. 75$           |
| $\frac{1}{1583427} = 544. 80$ its $\frac{1}{3}$     | $= 41. 91$            |
| $\frac{1}{14248845} = 60. 53$ its $\frac{1}{3}$     | $= 4. 03$             |
| $\frac{1}{128239605} = 6. 72$ its $\frac{1}{3}$     | $= 40$                |
| $\frac{1}{1154156445} = 74$ its $\frac{1}{3}$       | $= 4$                 |

The Log. of 2 in *Brigg's* form is — 301. 029. 995. 66

Again, let it be required to find *Brigg's* Logarithm of 3.

Because  $a=1, b=3, d=2$ , and  $z=4$ ; therefore  $\frac{d}{z} = \frac{1}{2}$ .  
And

And  $\frac{R}{2} + \frac{1}{3} \times \frac{R}{8} + \frac{1}{5} \times \frac{R}{32}$ , &c. will be the Logarithm of 3; wherefore, if R be divided by 4, and each quotient successively by the same number, and those again by the indices of the odd powers, the sum of these last quotients will be the *Brigg's* Logarithm of 3; but because this series converges very slow, the same Logarithm of 3 may be found much quicker, by finding the difference between the Logarithm of 2 already known, and the Logarithm of 3 required.

Put therefore  $a=2$ , and  $b=3$ , then will  $z=5$ , and  $d=\frac{1}{5}$ , and  $\frac{d^2}{z^2}$ , the common multiplicator equal to  $\frac{1}{25}$ , wherefore if R be divided by 5, and that quotient by 25, &c. and these several quotes by 1, 3, 5, 7, &c. the sum of these last quotients added to the Logarithm of 2, will give the Logarithm of 3.

## Operation,

|                               |                                   |     |                   |
|-------------------------------|-----------------------------------|-----|-------------------|
| $\frac{1}{5} R =$             | 173. 717. 792. 76                 | $=$ | 173. 717. 792. 76 |
| $\frac{1}{25} \text{ ---}$    | 6. 948. 711. 71 its $\frac{1}{5}$ | $=$ | 2. 316. 237. 24   |
| $\frac{1}{125} \text{ ---}$   | 277. 948. 47 its $\frac{1}{5}$    | $=$ | 55. 589. 69       |
| $\frac{1}{625} \text{ ---}$   | 11. 117. 94 its $\frac{1}{5}$     | $=$ | 1. 588. 28        |
| $\frac{1}{3125} \text{ ---}$  | 444. 73 its $\frac{1}{5}$         | $=$ | 49. 41            |
| $\frac{1}{15625} \text{ ---}$ | 17. 79 its $\frac{1}{5}$          | $=$ | 1. 62             |
| $\frac{1}{78125} \text{ ---}$ | 71 its $\frac{1}{5}$              | $=$ | 5                 |

Diff. between the Log. of 2 and 3 = 176. 091. 259. 05

Wherefore if to the Logarithm of 2 = 301. 029. 955. 66

the difference just found, be added, we shall have the Logarithm of 3 = 477. 123. 254. 71.

Again let it be required to find the Log. of 7.

Put  $a=6$ ,  $b=7$ , then will  $z=13$ , and  $d=\frac{1}{13}$ , whence

$\frac{d^2}{z^2} = \frac{1}{169}$ , and therefore  $\frac{z}{d} \times R = 66. 814. 535. 68$ , consequently.

|                                 |                                 |     |                  |
|---------------------------------|---------------------------------|-----|------------------|
| $\frac{1}{169} \text{ ---}$     | 66. 814. 535. 68                | $=$ | 66. 814. 535. 68 |
| $\frac{1}{2197} \text{ ---}$    | 395. 352. 28 its $\frac{1}{13}$ | $=$ | 131. 784. 09     |
| $\frac{1}{28541} \text{ ---}$   | 2. 339. 36 its $\frac{1}{13}$   | $=$ | 476. 87          |
| $\frac{1}{371033} \text{ ---}$  | 13. 84 its $\frac{1}{13}$       | $=$ | 1. 98            |
| $\frac{1}{4823429} \text{ ---}$ | 8 its $\frac{1}{13}$            | $=$ | 1                |

The diff. of the Log. 6 and 7 = 66. 946. 789. 63

Wherefore

To the Log. of 6 = 778. 151. 259. 38

Add the diff. found = 66. 946. 789. 63

The sum is the Log. of 7 = 845. 098. 049. 01

The



The Logarithm of all composite numbers, that is, all such as arise from the multiplication of others called factors, are easily found by the addition of the Logarithms of those factors; Thus the Log. of  $4 = \text{Log. } 2 + \text{Log. } 2$ , and the Log.  $10 = \text{Log. } 2 + \text{Log. } 5$ , and therefore the Log. of  $5 = \text{Log. } 10 - \text{Log. } 2$ ; the Log.  $6 = \text{Log. } 2 + \text{Log. } 3$ ; Log.  $8 = \text{Log. } 2 + \text{Log. } 4$ ; and the Log.  $9 = \text{Log. } 3 + \text{Log. } 3$ . Therefore from the Logarithms of the prime numbers 2, 3, and 7, which we have already found, all the Logarithms of the numbers under 11, may be computed by addition and subtraction only. And after the same manner may the whole cannon be constructed, the prime numbers being easily computed by the Theorems we have laid down.

The excellence of this method will be the more conspicuous if we compare the foregoing computations of the prime numbers with each other; for then it will appear, that as the prime number itself increases, the operation decreases, and the fewer steps are requisite; till at last the first step will be sufficient to find the difference to any number of places requisite.

Thus in the common tables, which extend but to seven places in decimals, when the difference  $d$  becomes equal to the one two hundredth part of the sum  $z$ , the first step,  $\frac{d}{z} \times R$ , will give the difference between the Logarithms true to the same number of places.

For if 868.588.9 be divided by 201, the sum of the numbers 100 and 101, the quotient 0043213, will give the difference between the Log. of 100 and 101, true to 7 places, which therefore being added to 2.0000000, will give 2.0043213, for the Logarithm of 101; so that the Logarithms of the prime numbers under 100 being found, the Logarithms of all the prime numbers above 100 may be found by one single division, and the Logarithms of any intermediate numbers by addition and subtraction only.

As the Logarithms 1, 10, 100, 1000, 10000, &c. in Brigg's form are 0, 1, 2, 3, 4, &c. it is easy to conceive that the Logarithms of all numbers which increase or decrease in a tenfold proportion, differ from each other only in their indices or characterestics, the fractional number remaining constantly the same: And hence it is that the fractional number .30102999566, which is the Logarithm of 2, when the index is a cypher, becomes the Logarithm of 20 when the index is 1, of 200 when it is 2, of 2000, when

59.38  
89.63  
40.01  
The





Theorem in page 360) when  $n$  is finite, when  $n$  becomes

infinte  $\frac{1}{1+L}$  will be equal to  $1 + nL + \frac{1}{2} n^2 L^2 + \frac{1}{6}$

$n^3 L^3 + \frac{1}{24} n^4 L^4 + \frac{1}{120} n^5 L^5 + \frac{1}{720} n^6 L^6 + \frac{1}{5040}$

$n^7 L^7, \&c. = 1 + x$ . Also becaufe  $\frac{1}{n} \times x + \frac{1}{2} x^2 +$

$\frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{5} x^5 + \frac{1}{6} x^6 \&c. = 1 - \frac{1}{1-x}$  is the

Logarithm of the ratio of 1 to  $1-x$ , therefore  $1 - \frac{1}{1-x}$

$= L$ , and consequently  $1 - x = \frac{1}{1-L}$ , and  $1 - x =$

$1 - L$ , wherefore by the same theorem  $1 - x = 1 - L$ , will

be equal to  $1 - nL + \frac{1}{2} n^2 L^2 - \frac{1}{6} n^3 L^3 + \frac{1}{24} n^4 L^4 -$

$\frac{1}{120} n^5 L^5 + \frac{1}{720} n^6 L^6 - \frac{1}{5040} n^7 L^7, \&c.$

Whence  $1 + x = 1 + nL + \frac{1}{2} n^2 L^2 + \frac{1}{6} n^3 L^3 + \frac{1}{24}$

$n^4 L^4 + \frac{1}{120} n^5 L^5, \&c.$  will be a general Theorem for

finding the number from the Logarithm given, of any form

whatsoever.

And becaufe  $n = 1000000, \&c.$  in *Neper's* form,  $1 + x$

$= 1 + L + \frac{1}{2} L^2 + \frac{1}{6} L^3 + \frac{1}{24} L^4 + \frac{1}{120} L^5, \&c.$  will

give the number answering to any Logarithm in *Neper's* form.

Hence any one term of the ratio, whereof  $L$  is the Loga-

rithm being given, the other term will be readily found;

for by putting  $a$  for the lesser of the two terms, and  $b$  for

the greater,  $a \times 1 + L + \frac{1}{2} L^2 + \frac{1}{6} L^3 + \frac{1}{24} L^4 + \frac{1}{120}$

$L^5, \&c.$  will give the greater, and  $b \times 1 - L + \frac{1}{2} L^2 - \frac{1}{6}$

$L^3 + \frac{1}{24} L^4 - \frac{1}{120} L^5, \&c.$  will give the lesser, in *Neper's*

form.

A 2

Or,

Or,

$$a \times 1 + nL + \frac{1}{2}n^2L^2 + \frac{1}{6}n^3L^3 + \frac{1}{24}n^4L^4 + \frac{1}{120}n^5L^5, \text{ \&c.} = b : \text{ and } a \times 1 - nL + \frac{1}{2}n^2L^2 - \frac{1}{6}n^3L^3 + \frac{1}{24}n^4L^4 - \frac{1}{120}n^5L^5, \text{ \&c.} = a \text{ in any form.}$$

Let  $d$  = the difference between the given Logarithm and the next nearest tabular Logarithm; then will  $a \times 1 + d + \frac{1}{2}d^2 + \frac{1}{6}d^3 + \frac{1}{24}d^4 + \frac{1}{120}d^5, \text{ \&c.} = N$ . Or,

$$b \times 1 - d + \frac{1}{2}d^2 - \frac{1}{6}d^3 + \frac{1}{24}d^4 - \frac{1}{120}d^5, \text{ \&c.} = N, \text{ the number required in Neper's form. Or } a \times 1 + nd + \frac{1}{2}n^2d^2 + \frac{1}{6}n^3d^3 + \frac{1}{24}n^4d^4 + \frac{1}{120}n^5d^5, \text{ \&c.} = N.$$

And  $b \times 1 - nd + \frac{1}{2}n^2d^2 - \frac{1}{6}n^3d^3 + \frac{1}{24}n^4d^4 - \frac{1}{120}n^5d^5, \text{ \&c.} = N$ , the correspondent number in Logarithms of any species.

Let it, for example, be required to find the number answering to 7.5713740280 in *Brigg's* form.

Because 7.5713710453, the nearest tabular number to this is less than the given number, put 37271000, the number answering to it =  $a$ , and 000029827, the difference between the two Logarithms =  $d$ , then will  $n = 230258, \text{ \&c.}$  multiplied by 37271000, the next nearest number in the table, give 0000068683; which being increased by 1, and multiplied by 37271000, the next nearest number in the table, will give 37271255.988 for the number answering to the given Logarithm: Or, if 0000068683 be multiplied by 37271000 =  $a$ , the product 255.998, added to  $a = 37271000$  will give 37271255.988 for the number required, the same as before.

Hence, and from the general Theorem, may the number answering to any Logarithm be found, and consequently an anti-logarithmic canon be constructed, shewing the natural numbers answering to every Logarithm; whence the numbers answering to any Logarithm might be found with the same ease as we find the Logarithm of any number in the common Logarithmic Cannon. Such a Cannon has been lately published by the ingenious Mr. *Dodson*, which has rendered the Logarithmic Tables compleat. Having



Having thus shewn the method of constructing the Logarithmic Tables, we shall now proceed to shew how the Logarithmic fines, tangents and secants are computed; but it will be requisite previously to explain the manner how the natural sine, tangent, secant, &c. of any arch may be found. In order to which let C (Plate IX. Fig. 19) be the center of a circle,  $CA = CS = r$  the radius,  $AS = a$  any arch thereof,  $Sr = s$  its right sine; then will  $Cr$ , the co-sine  $= \sqrt{1 - ss}$ , by the 47.1.

Let CM be another radius of the circle, infinitely near to CS; then will  $Mn = \dot{s}$  be the fluxion of the sine  $Sr$ , and the infinitely small arch  $MS = \dot{a}$ , the fluxion of the arch AS. And because the triangles  $CSr$  and  $MnS$  are similar, it will be as  $Cr : CS :: Mn : SM$ ; that is  $\sqrt{1 - ss} : 1 :: \dot{a} : \dot{s}$ ; wherefore  $\dot{a} = \frac{\dot{s}}{\sqrt{1 - ss}}$ , that is, if the fluxion of the right-sine be divided by the square root of  $1 - ss$ , the integral, or flowing quantity of the quotient will be the arch itself.

The square root of  $1 - ss$ , being extracted will be  $1 - \frac{1}{2}s^2 - \frac{1}{8}s^4 - \frac{1}{16}s^6$ , &c. by which the fluxion  $\dot{s}$  must be divided, which will give the fluxion of the arch SA, as follows,

$$\left. \begin{array}{l} 1 - \frac{1}{2}s^2 - \frac{1}{8}s^4 - \frac{1}{16}s^6, \text{ \&c.} \end{array} \right\} \begin{array}{l} : (s + \frac{1}{2}s^3 + \frac{3}{8}s^5 + \frac{5}{16}s^7, \text{ \&c.} \\ : - \frac{1}{2}s^3 - \frac{1}{8}s^5 - \frac{1}{16}s^7, \text{ \&c.} \\ \hline + \frac{1}{2}s^3 + \frac{1}{8}s^5 + \frac{1}{16}s^7, \text{ \&c.} \\ + \frac{1}{2}s^3 - \frac{1}{8}s^5 - \frac{1}{16}s^7, \text{ \&c.} \\ \hline + \frac{3}{8}s^5 + \frac{1}{8}s^7, \text{ \&c.} \\ + \frac{3}{8}s^5 - \frac{3}{16}s^7, \text{ \&c.} \\ \hline + \frac{5}{16}s^7, \text{ \&c.} \end{array}$$

A 22

Consequently

Consequently  $s + \frac{1}{2}s^2 + \frac{3}{8}s^4 + \frac{5}{16}s^6, \&c. = a$ ,  
the fluxion of the arch, whose fluent or flowing quantity is  
 $s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{5}{112}s^7, \&c.$

Therefore  $a = s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{5}{112}s^7, \&c.$  or  $a =$   
 $s + \frac{1 \times 1}{2 \times 3}s^3 + \frac{1 \times 1 \times 3 \times 3}{2 \times 3 \times 4 \times 5}s^5 + \frac{1 \times 1 \times 3 \times 3 \times 5 \times 5}{2 \times 3 \times 4 \times 5 \times 6 \times 7}s^7, \&c.$

Hence we may find the length of any arch of a circle, the  
right-line of that arch being given.

And because the radius or semidiameter of a circle is  
equal to the side of a hexagon inscribed in a circle (*Cor. 1.*  
*Prop. 15. Book 4.*) or chord of 60 degrees, therefore the sine  
of 30 degrees will be equal to half the radius: And conse-  
quently when the radius of a circle is given, the sine of 30  
degrees is given also.

Let it therefore be required to find the length of the arch  
of thirty degrees, to ten decimal places, which is abundant-  
ly sufficient for any common purpose, the radius being = 1.

Because the radius = 1, the sine of 30 deg. =  $s$ , will be  
=  $\frac{1}{2}$ , and consequently its square  $\frac{1}{4}$ , its cube  $\frac{1}{8}$ , &c. And  
as to multiply by  $\frac{1}{4}$ , and divide by 4, is the same thing, it  
follows that if the right-line =  $\frac{1}{2}$  = .5 be divided by 4,

and the several quotients resulting from thence be multiplied  
by 1,  $\frac{1}{6}$ ,  $\frac{3}{40}$ , &c. the sum of these products will be the  
length of the arch required.

|          |   |                  |     |                              |                  |
|----------|---|------------------|-----|------------------------------|------------------|
| $s$      | = | .5000, 0000, 000 | —   | —                            | .5000, 0000, 000 |
| $s^3$    | = | 1250, 0000, 000  | its | $\frac{1}{8}$                | 208, 3333, 333 + |
| $s^5$    | = | 312, 5000, 000   | its | $\frac{3}{40}$               | 23, 4375, 000    |
| $s^7$    | = | 78, 1250, 000    | its | $\frac{5}{112}$              | 3, 4877, 232 +   |
| $s^9$    | = | 19, 5312, 500    | its | $\frac{7}{256}$              | 5933, 974 —      |
| $s^{11}$ | = | 4, 8828, 125     | its | $\frac{63}{2684}$            | 1092, 391 —      |
| $s^{13}$ | = | 1, 2207, 031     | its | $\frac{231}{1048576}$        | 211, 826 —       |
| $s^{15}$ | = | 3851, 758        | its | $\frac{143}{3435973}$        | 42, 617 +        |
| $s^{17}$ | = | 762, 939         | its | $\frac{643}{7783216}$ , &c.  | 8, 813 +         |
| $s^{19}$ | = | 190, 735         | its | $\frac{131}{1270080}$ , &c.  | 1, 863 —         |
| $s^{21}$ | = | 47, 684          | its | $\frac{461}{1000000}$ , &c.  | 400 +            |
| $s^{23}$ | = | 11, 921          | its | $\frac{679}{8192000}$ , &c.  | 87 +             |
| $s^{25}$ | = | 2, 980           | its | $\frac{1301}{5570000}$ , &c. | 19 +             |
| $s^{27}$ | = | 745              | its | $\frac{244}{3125000}$ , &c.  | 4 +              |

The length of the arch of 30 deg. = 5235, 9877, 559

If

If this arch be multiplied by 6, we shall have the length of the arch of the semicircle when the radius is 1, or the length of the whole circumference when the diameter is 1.

$$\begin{array}{r} .5235, 9877, 559 \\ \times 6 \\ \hline 3.1415, 9265, 354 \end{array}$$

The absolute necessity of finding, as near as possible, the proportion between the diameter and circumference of a circle, for exactly true it never can be found, has engaged some of the greatest Mathematicians to approximate as near as possible to the truth. *Van Ceulen* carried it to 32 decimal places, which he ordered to be engraved on his Tomb-stone, as a memorial of so great a work. His numbers are as follows,

If the diameter be 1.0000000, &c. the circumference will be 3.141592,653589,793238,462643,383279,502884.

Since *Van Ceulen*, the indefatigable Mr. *Abraham Sharp*, has carried it to twice the number of places, and shewn us that if the diameter be 1.0000000, &c. the circumference will be 3.141592,653589,793238,462643,383279,502884,197169,399375,105820,974944,592307,816405,2.

And the learned Mr. *John Machin*, has carried it to 100 places, which is a degree of exactness far exceeding all imagination. And is as follows.

If the diameter of a circle be 1,000, &c. the circumference will be 3.141592,653589,793238,462643,383279,502884,197169,399375,105820,974944,592307,816405,286208,998628,034825,342117,0679 of the same parts.

Because  $a = s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{5}{11}s^7$ , &c. Therefore,

$$s = a - \frac{1}{6}a^3 + \frac{1}{120}a^5 - \frac{1}{5040}a^7, \text{ \&c.}$$

Or,

$$s = a - \frac{1}{2 \times 3}a^3 + \frac{1}{2 \times 3 \times 4 \times 5}a^5 - \frac{1}{2 \times 3 \times 4 \times 5 \times 6}a^7, \text{ \&c.}$$

For put  $s = Aa + Ba^3 + Ca^5$ , &c.

$$\text{Then } \frac{1}{6}s^3 = + \frac{1}{6}A^3a^3 + \frac{1}{2}A^2Ba^5, \text{ \&c.}$$

$$\text{And } \frac{3}{40}s^5 = + \frac{3}{40}A^5a^5, \text{ \&c.}$$

A a 3

Consequently



Consequently  $A a = a$ , and  $A = 1$ ; also  $B + \frac{1}{6} A^3 = 0$ , and  $B = -\frac{1}{6}$ . Also  $C + \frac{1}{2} A^2 B + \frac{3}{40} A^5 = 0$ . And  $C = -\frac{1}{2} A^2 B - \frac{3}{40} A^5 = -\frac{1}{2} \times -\frac{1}{6} - \frac{3}{40} = +\frac{1}{120}$ ; wherefore  $A = 1$ ,  $B = -\frac{1}{6}$ ,  $C = +\frac{1}{120}$  &c. wherefore  $s = a - \frac{1}{6} a^3 + \frac{1}{120} a^5$ , &c. Q. E. D.

Let it be required to find the natural sine of 10 degrees.

Because the length of the semi-circumference is 3.1415,

9265, 354, therefore the length of 10 deg. or  $\frac{1}{18}$  will be .1745, 3292, 5196.

Put therefore  $a =$  1745, 3292, 519

Then will  $-\frac{1}{6} a^3 =$  8, 8609, 615

And  $a - \frac{1}{6} a^3 =$  1736, 4682, 904

Again  $+\frac{1}{120} a^5 =$  134, 960

And  $a - \frac{1}{6} a^3 + \frac{1}{120} a^5 =$  1736, 4817, 864

Also  $-\frac{1}{5040} a^7 =$  -97

And  $a - \frac{1}{6} a^3 + \frac{1}{120} a^5 - \frac{1}{5040} a^7 =$  1736, 4817, 767

Therefore  $s = .1736, 4817, 767$ , the sign of 10 degrees required.

The sine of an arch being given, its co-sine may be easily found; for because  $CS_q = SR_q + CR_q$  (by 47.1) see Plate IX. Fig. 3. therefore  $\sqrt{CS_q - SR_q} = CR$ . That is, if from the square of the radius, be taken the square of the right sine, the square root of the remainder will be the co-sine required.

Therefore if from the square of radius  $= 1$ , be taken the square root of the right-sine  $= a - \frac{1}{6} a^3 + \frac{1}{120} a^5$ , &c. the square root of the remainder will be the co-sine required.

Thus,

Thus,

$$s = a - \frac{1}{6} a^3 + \frac{1}{120} a^5, \&c.$$

$$s = a - \frac{1}{6} a^3 + \frac{1}{120} a^5, \&c.$$

$$a^2 - \frac{1}{6} a^4 + \frac{1}{120} a^6, \&c.$$

$$- \frac{1}{6} a^4 + \frac{1}{120} a^6, \&c.$$

$$+ \frac{1}{120} a^6, \&c.$$

$ss = a^2 - \frac{1}{3} a^4 + \frac{1}{45} a^6, \&c.$  which taken from the square of radius  $= 1$ , leaves  $1 - a^2 + \frac{1}{3} a^4 - \frac{1}{45} a^6, \&c.$  the square root of which will be the co-sine required.

$$1 - aa + \frac{1}{3} a^4 - \frac{1}{45} a^6, \&c. (1 - \frac{1}{2} a^2 + \frac{1}{24} a^4 - \frac{1}{720} a^6, \&c.$$

$$2 - \frac{1}{2} aa) - aa + \frac{1}{3} a^4$$

$$- aa + \frac{1}{3} a^4$$

$$2 - aa, \&c.) + \frac{1}{12} a^4 - \frac{1}{45} a^6, \&c.$$

$$+ \frac{1}{12} a^4 - \frac{1}{45} a^6, \&c.$$

$$2 - aa, \&c.) - \frac{1}{360} a^6, \&c.$$

$$- \frac{1}{360} a^6, \&c.$$

Wherefore the co-sine  $= 1 - \frac{1}{2} a^2 + \frac{1}{24} a^4 - \frac{1}{720} a^6, \&c.$

Or the co-sine  $= 1 - \frac{1}{1 \times 2} a^2 + \frac{1}{1 \times 2 \times 3 \times 4} a^4 -$

$$\frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6} a^6, \&c.$$

The sines being thus obtained, the tangents and secants are easily found as follows.

For because the triangles CRS, CAT, (Plate IX. Fig. 3.) are similar, it will be,

As CR, the co-sine, is to RS, the right-sine; so is CA, the radius, to AT, the tangent.

And because the radius  $= 1$ , therefore the quotient of the right-sine divided by the co-sine will give the tangent of the same arch.

Hence the tangent series are easily found; for if  $a - \frac{1}{6} a^3 + \frac{1}{120} a^5, \&c.$  the right-sine, be divided by  $1 - \frac{1}{2} aa + \frac{1}{24} a^4, \&c.$  the co-sine, the quotient will give the tangent required, as in the following example.

$$\begin{array}{r}
 1 - \frac{1}{2}aa + \frac{1}{24}a^4, \&c.) \quad a - \frac{1}{2}a^3 + \frac{1}{120}a^5, \&c. \quad (a + \frac{1}{3}a^3 + \frac{2}{15}a^5, \&c. \\
 \quad \quad \quad a - \frac{1}{2}a^2 + \frac{1}{24}a^4, \&c. \quad \quad \quad (a^5, \&c. \\
 \quad \quad \quad \frac{1}{3}a^3 + \frac{1}{30}a^5, \&c. \\
 \quad \quad \quad \frac{1}{3}a^3 - \frac{1}{3}a^5, \&c. \\
 \quad \quad \quad \hline
 \quad \quad \quad + \frac{2}{15}a^5, \&c.
 \end{array}$$

Hence if  $a$  = the length of any arch, the correspondent tangent will be  $a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 + \frac{62}{2835}a^9 + \frac{1382}{52935}a^{11}, \&c.$

Again, because the triangles ACT, GBC, are similar, it will be, as AT, the tangent, to AC, radius; so is CB, radius, to BG, the co-tangent.

Hence the radius is a mean proportional between the tangent and co-tangent of an arch, and because the radius = 1, if 1 be divided by  $a + \frac{1}{3}a^3 + \frac{2}{15}a^5, \&c.$  the tangent, the

quotient  $\frac{1}{a} - \frac{1}{3}a + \frac{1}{45}a^3 - \frac{2}{945}a^5 + \frac{1}{4725}a^7 - \frac{2}{93555}a^9, \&c.$  will be the series for finding the co-tangent, from the arch first given.

Again, because the triangles CRS, CAT, are similar, it will be as CR, the co-sine, to CS, the radius; so is CA, the radius, to CT, the secant.

Hence the radius is a mean proportional between the secant and co-sine of an arch.

And therefore if 1 be divided by  $1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6, \&c.$  the co-sine, the quotient  $1 + \frac{1}{2}a^2 + \frac{1}{24}a^4 + \frac{1}{720}a^6 + \frac{17}{8064}a^8 + \frac{5}{3840}a^{10}, \&c.$  will be the series for finding the secant from the arch first given.

A cannon of natural sines, tangents, and secants being constructed according to the foregoing examples, it will be easy to find the corresponding artificial sines, tangents, and secants; for the Logarithms of the former will be the latter.

But the Logarithmic sine, tangent, or secant of any arch may be found, independantly of the tables of Logarithms;

for by putting  $s$  for the sine of any arch, and  $q = \frac{s-1}{s+1}$  we

shall have  $\frac{1}{n} \times q + \frac{1}{3}q^3 + \frac{1}{5}q^5 + \frac{1}{7}q^7, \&c.$  for the Logarithmic sine of the same arch, and if instead of  $s$ , in the former series, we put  $t$  for the tangent, or  $c$  for the secant of any arch, the same series will produce the Logarithmic tangent or secant of the given arch.

Or if instead of  $s$ , the right-sine, in the above series we put  $a$ , for the length of the arch itself then will  $\frac{1}{n} \times n - \frac{1}{2}a^2 - \frac{1}{12}a^4 - \frac{1}{45}a^6 - \frac{17}{1296}a^8 - \frac{1}{14175}a^{10}, \&c.$  be the Logarithmic



Logarithmic co-sine of the same arch, in *Brigg's* form, the radius being 10.0000000000, and  $n = 2.3025,8509,29$ .

We have already shewn that if the radius of a circle be put equal to 1, the length of the arch of 10 degrees will be equal to 1745,3292,519, of the same parts; put this therefore equal to  $a$ .

$$\begin{array}{lcl} \text{Then will } \frac{1}{2} a^2 & = & \text{0152,3087,10} \\ \text{And } \frac{1}{12} a^4 & = & \text{7732,65} \\ \text{Also } \frac{1}{45} a^5 & = & \text{62,81} \\ \text{And } \frac{1}{2520} a^8 & = & \text{58} \\ \text{Therefore } \frac{1}{2} a^2 + \frac{1}{12} a^4 + \frac{1}{45} a^5 + \frac{1}{2520} a^8 & = & \text{0153,0883,14} \end{array}$$

This value being taken from  $n = 2.3025,8509,29$ , will leave 2,2872,7625,15, which being multiplied by 4342,9448,190, will give the fractional part of the Logarithmic sine of 80 degrees = .9933,5145,89, to which prefixing 9 for an index, because the radius is equal to 10, will give 9.9933,5145,89, for the Logarithmic sine of 80 degrees in *Brigg's* form; and after the same manner may the sine of any arch be found independant of the Logarithmic tables, and consequently the whole table of sines constructed.

Having thus computed a table of Logarithmic sines, the Logarithmic tangents and secants may be had by common addition and subtraction only. For,

1. As the co-sine of an arch, is its right sine; so is the radius, to the tangent of the same arch.

Therefore if to the right-sine of an arch we add the radius, and from that sum subtract the co-sine of the same arch, the remainder will be the tangent required.

$$\begin{array}{lcl} \text{Thus, To the Log. sine of 80 degrees} & = & 9.99335,14589 \\ \text{Add the radius} & = & 10.0000000000 \\ & & \text{19.99335,14589} \end{array}$$

$$\text{And subtract the co-sine of 80 degrees} = 9.23967,02300$$

$$\text{The remainder is the Log. tang. of 80 deg.} = 10.75368,12289$$

2. As the co-sine, is to radius; so is radius to the secant of the same arch.

Therefore, from twice radius subtract the co-sine, and the remainder will give the secant of the same arch.

$$\begin{array}{lcl} \text{Thus, from twice radius} & = & 20.0000000000 \\ \text{Take the co-sine of 10 degrees} & = & 9.99335,14589 \\ \text{The remainder is the secant of 10 deg.} & = & 10.00664,85411 \end{array}$$

Hence it appears that if the Logarithmic sine and co-sine of an arch be known, its tangent and secant may be readily found; and consequently the whole cannon easily computed.

The

The ingenious Mr. *Symson*, in his *Trigonometry*, page 47 & seq. has given us a very compendious method of finding the Logarithms of large numbers, one from another, by addition and subtraction only, which is as follows.

1. Let A, B, C, denote any three numbers in arithmetical progression, not less than 10000 each, whereof the common difference is 100.

2. From twice the Logarithm of B, subtract the sum of the Logarithms of A and C, and let the remainder be divided by 10000.

3. Multiply the quotient by 49.5, and to the product add  $\frac{1}{1000}$  part of the difference of the Logarithms of A and B; then the sum will be the excess of the Logarithm of A + 1 above that of A.

4. From this excess let the quotient (found by Rule 2.) be continually subtracted, that is, first from the excess itself, then from the remainder, then from the next remainder, &c. &c.

5. To the Logarithm of A add the said excess, and to the sum add the first of the remainders; to the last sum add the next remainder, &c. &c. then the several sums, thus arising, will exhibit the Logarithms of A + 1, A + 2, A + 3, &c. respectively.

Thus let it be proposed to find the Logarithms of all the whole numbers between 17900 and 18100; those of the two extremes 17900 and 18100, and that of the mean (18000) being given.

Then the Logarithm of  $\begin{Bmatrix} A \\ B \\ C \end{Bmatrix}$  being equal to  $\begin{Bmatrix} 4.252853031 \\ 4.255272505 \\ 4.257678575 \end{Bmatrix}$   
 we shall have  $\frac{2 \text{ Log. B} - \text{Log. A} - \text{Log. C}}{10000} = .00000000134$

(see Rule 2.) which multiplied by 49.5, and the product added to  $\frac{\text{Log. B} - \text{Log. A}}{100}$  gives .00002426107 for the excess of the Logarithm of A + 1 above that of A (by Rule 3.) From whence the work, being continued according to Rule 4 and 5, will stand as follows.

17900  
17910  
17920  
17930  
17940  
17950  
17960  
17970  
17980  
17990  
18000  
18010  
18020  
18030  
18040  
18050  
18060  
18070  
18080  
18090  
18100



|         |                             |      |                |                |
|---------|-----------------------------|------|----------------|----------------|
| 1000024 | 25107 excess                | 4.25 | 2853031        | Log. of 17900. |
|         | 134                         |      | 2426107 excess |                |
|         | 25973 1 <sup>st</sup> Rem.  |      | 287729207      | Log. of 17901. |
|         | 134                         |      | 2425973        |                |
|         | 25839 2 <sup>d</sup> Rem.   |      | 290155280      | Log. of 17902. |
|         | 134                         |      | 2425839        |                |
|         | 25705 3 <sup>d</sup> Rem.   |      | 292581019      | Log. of 17903. |
|         | 134                         |      | 2425705        |                |
|         | 25571 4 <sup>th</sup> Rem.  |      | 295006724      | Log. of 17904. |
|         | 134                         |      | 2425571        |                |
|         | 25437 5 <sup>th</sup> Rem.  |      | 297432295      | Log. of 17905. |
|         | 134                         |      | 2425437        |                |
|         | 25303 6 <sup>th</sup> Rem.  |      | 299857732      | Log. of 17906. |
|         | 134                         |      | 2425303        |                |
|         | 25169 7 <sup>th</sup> Rem.  |      | 302283035      | Log. of 17907. |
|         | 134                         |      | 2425169        |                |
|         | 25035 8 <sup>th</sup> Rem.  |      | 304708204      | Log. of 17908. |
|         | 134                         |      | 2425035        |                |
|         | 24901 9 <sup>th</sup> Rem.  |      | 307133239      |                |
|         | 134                         |      | 2424901        |                |
|         | 24767 10 <sup>th</sup> Rem. |      | 309558140      | Log. of 17909. |
|         | &c. &c.                     |      | &c. &c.        |                |

*Note,* The Logarithms found by this method, in numbers between 10000 and 20000, are true to 8 or 9 places of figures. Those of numbers between 20000 and 50000 err only in the 9th or 10th place; and those above 50000 are true to ten places at least.

Having sufficiently explained the manner of constructing the tables of Logarithms; it remains that we shew the use of those tables.

1. We having already observed that addition of Logarithms answers to multiplication of natural numbers, and subtraction to division. Thus if it were required to multiply 8.5 by 10, the operation will be as follows,

|                     |       |             |
|---------------------|-------|-------------|
| To the Log. of 8.5. | _____ | = 0.9294189 |
| Add the Log. of 10  | _____ | = 1.0000000 |
| The Sum             | _____ | = 1.9294189 |

Which is the Log. of the product required, and the number corresponding thereto is 85.

Or if it were required to divide 9712.8 by 456, it will stand thus,

From



## An APPENDIX.

From the Log. of 9712.8 ————— = 3.9873444  
 Take the Log. of 456 ————— = 2.6589648  
 Remains ————— 1.3283796

Which is the Log. of the quotient required, and whose corresponding number is 21.3.

*To find the Complement of a Logarithm.*

Begin at the left-hand, and set down the difference between each figure and 9, except the first, which must be subtracted from ten. Thus you will find the complement of the Log. of 456, viz. of 2.6589648 to be 7.3410352.

*2. To perform the rule of Proportion by the Logarithms.*

Add the Logarithms of the second and third terms together, and subtract the Log. of the first from that sum; the remainder will be the Log. of the fourth term or number sought. Or instead of subtracting the Log. of the first term, add its complement, and the result will be the same: But observe that in adding a complement you must always cancel 1 in the tens place of the index, for every complement so added.

Thus, let it be required to find a fourth number in geometrical proportion to the three following, viz. 4, 9, 12.

To the Log. of 9 ————— = 0.9542425  
 Add the Log. of 12 ————— = 1.0791812  
 And from their sum ————— = 2.0334237  
 Take the Log. of 4 ————— = 0.6020600  
 The remainder is the Log. of 27 ————— = 1.4313637  
 So that the fourth number required is 27.

Or thus,

Complement of the Log. of 4 is ————— 9.3979400  
 Log. of ————— 9 ————— 0.9542425  
 Log. of ————— 12 ————— 1.0791812  
 Sum is Log. of 27 ————— 11.4313637

The same as before, except in the index, where 1 in the place of tens must be cancelled, as we before observed.

*Example 2.* Suppose 100*l.* in one year, or 365 days, gain 6*l.* interest; what will 5173*l.* gain in 321 days?

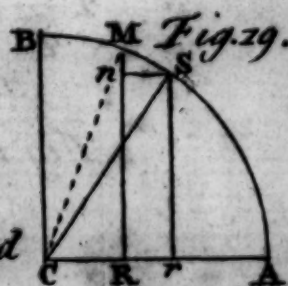
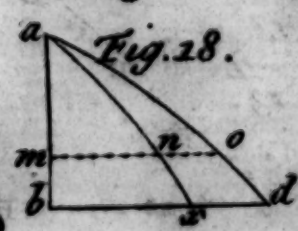
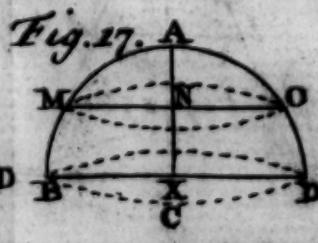
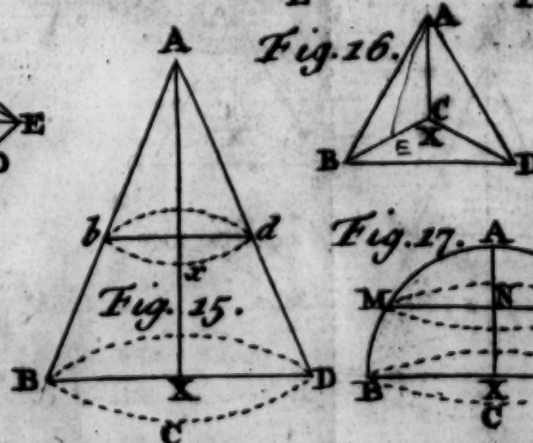
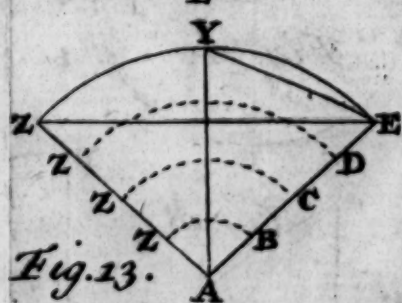
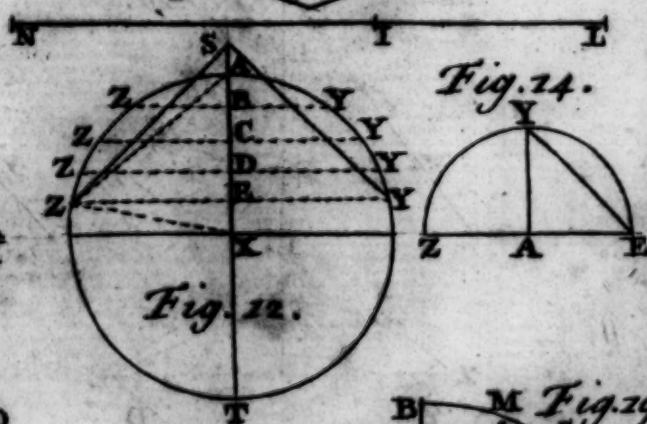
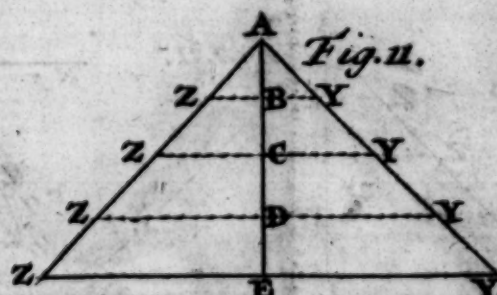
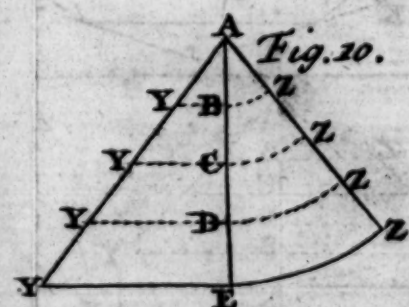
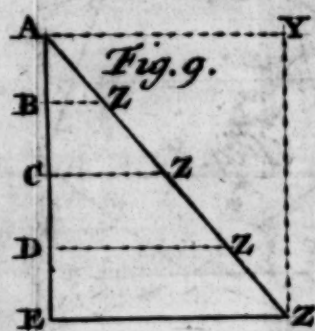
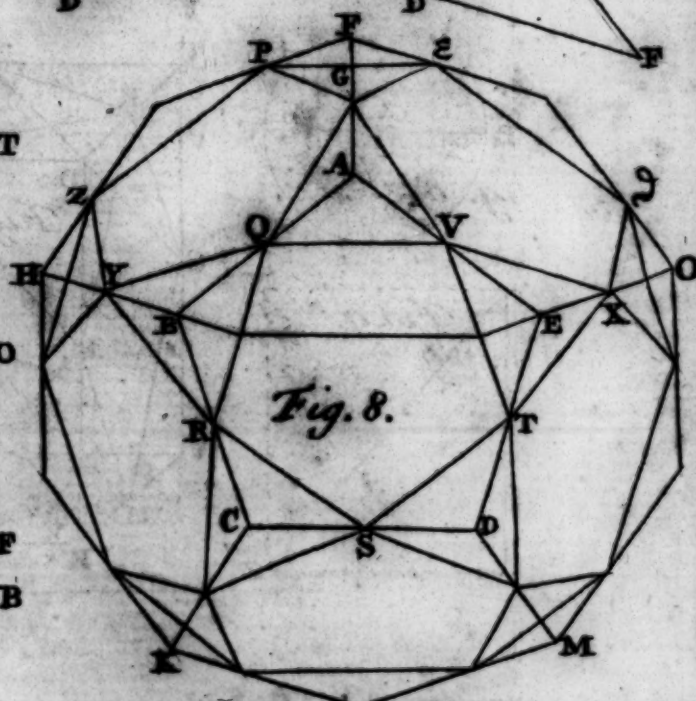
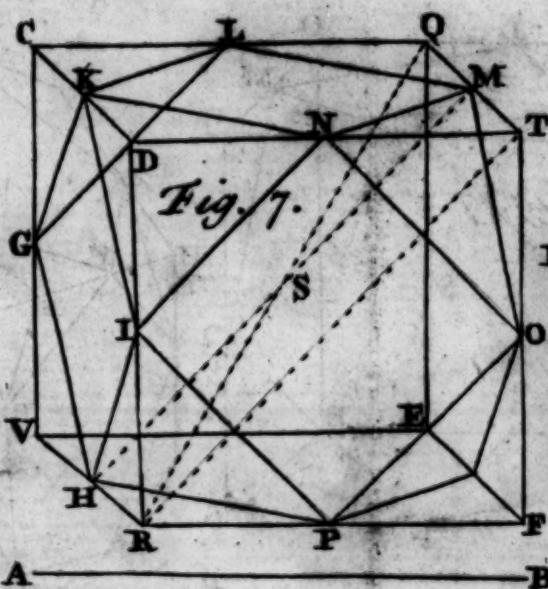
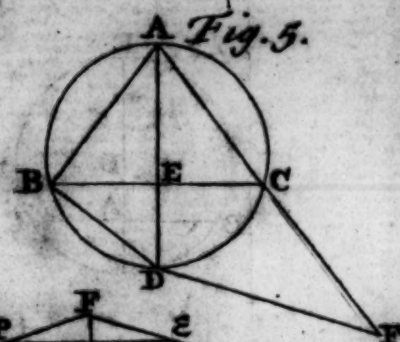
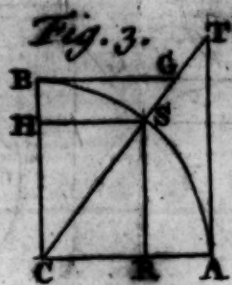
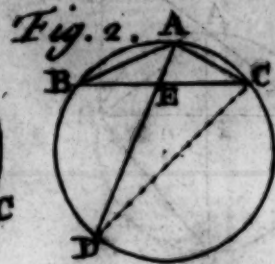
*Answer,* 272.9643*l.* or 272*l.* 19*s.* 3*d.*  $\frac{1}{2}$  nearly.

†

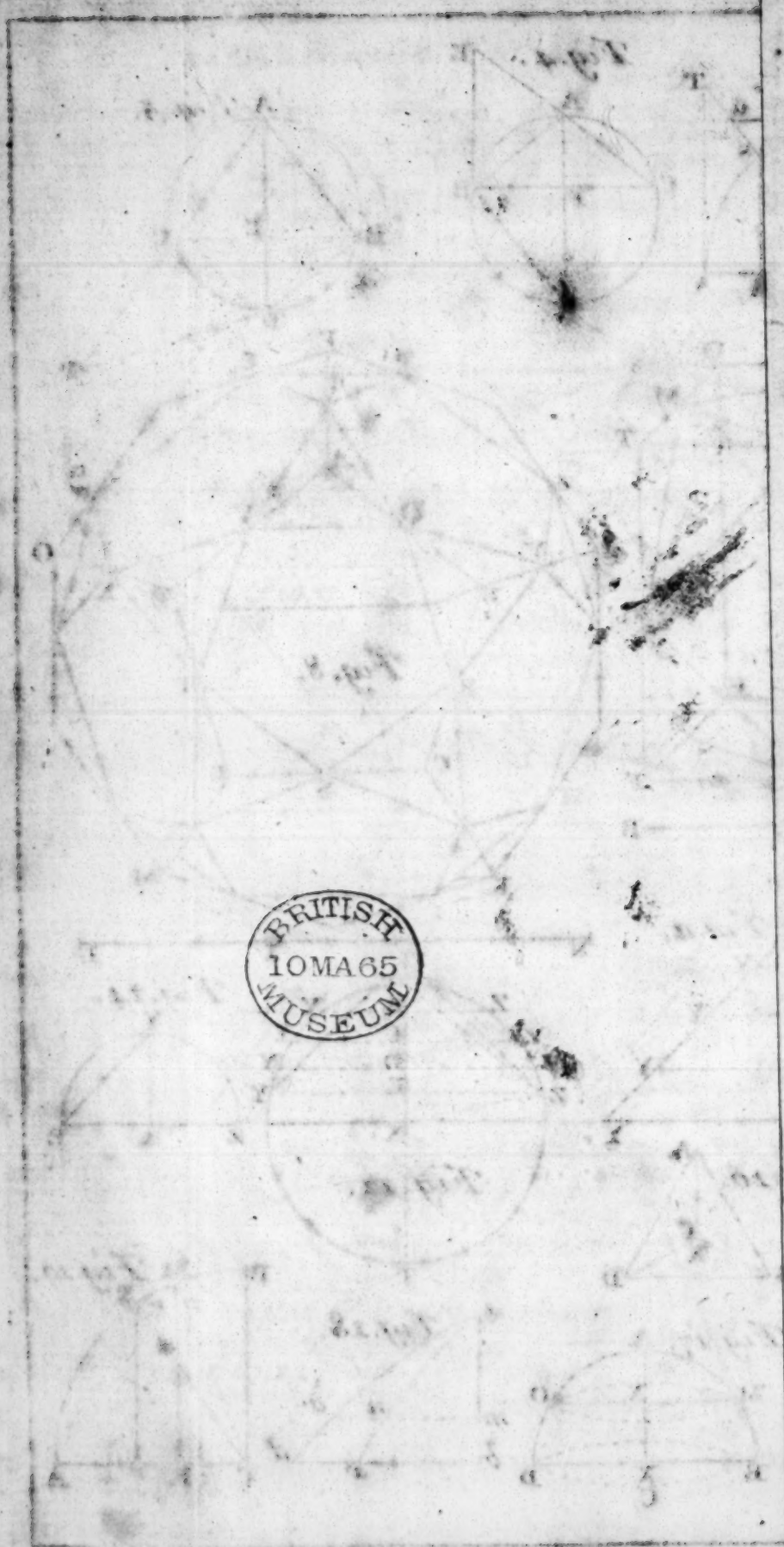
Comp.











|                          |          |            |
|--------------------------|----------|------------|
| Comp. of the Log. of 100 | _____    | 8.0000000  |
| Comp. of the Log. of 365 | _____    | 7.4377071  |
| Log. of _____            | 6        | 0.7781513  |
| Log. of _____            | 5173     | 3.7137425  |
| Log. of _____            | 321      | 2.5065050  |
| Log. of _____            | 272.9643 | 72.4361059 |

In the above operation, 2 in the place of tens in the index is cancelled, because two complements are added.

### 3. To raise Powers by Logarithms.

Multiply the Logarithm of the given number by the index of the required power, the product will be the Logarithm of the power sought.

*Example.* Let it be required to find the cube of 32 by the Logarithms.

The Log. of 32 is \_\_\_\_\_ 1.5051500  
 Which multiply by the index of the power \_\_\_\_\_ 3  
 The Log. of 32768 \_\_\_\_\_ 4.5154500  
 Which is the cube of 32 required.

*Example 2.* Let it be required to find the cube of 0.009 by the Logarithms.

The Log. of 0.009 is (see page 368) \_\_\_\_\_ 3.9542425  
 Which multiply by \_\_\_\_\_ 3  
 Which is the Log. of 0.000000729 \_\_\_\_\_ 7.8027275

*N.B.* When the Log. has a negative index, as in the above example, whatever is carried from the decimal part of the Log. must be considered as affirmative, and subtracted from the multiplication of the negative index; for the decimal part of every Log. is affirmative, whether the whole Log. taken together be so or not. Thus in the above example, the negative index 3 being multiplied by 3, produces 9, from which the 2 carried from the decimal part being subtracted, gives — 7 for the index of the Log. required.

*Example 3.* Let the 6th power of 0.0032 be required.

The Log. of 0.0032 is \_\_\_\_\_ 3.50515500  
 Which multiplied by \_\_\_\_\_ 6  
 The product is \_\_\_\_\_ 15.03093000  
 Which is the Log. of \_\_\_\_\_ 0.0000000000000010739

## AN APPENDIX.

## 4. To extract the Roots of Powers by the Logarithms.

Divide the Log. of the number by the index of the power, the quotient is the index of the root sought.

*Example.* Let it be required to extract the cube root of 6751269.

The Log. of 6751269 is 6.8293854, which being divided by 3, the index of the power, gives 2.2764618. the Log. of 189, the cube root sought.

*Example 2.* Let it be required to extract the root of the 6th power of 0.00000,00000,00000,00000,00000,00000,63 by the Logarithms.

The Log. of 0,00000,00000,00000,00000,00000,00000,63 is — 31.8061800, which being divided by 6, the index of the power gives — 5.3010300, which is the Log. of 0.00002, the root of the sixth power required.

And after the same manner may the root of any other power be extracted.

## 5. To find mean proportionals between any two numbers.

Subtract the Log. of the least term from the Log. of the greater, and divide the remainder by a number which exceeds by one the number of means sought; then add the quotient to the Log. of the lesser term, or subtract it from the Log. of the greatest, continually, and you will have the Logarithms of all the mean proportionals required.

*Example.* Let it be required to find three mean proportionals between 106 and 100.

|                        |                            |
|------------------------|----------------------------|
| Log. of 106            | 2.0253059                  |
| Log. of 100            | 2.0000000                  |
| Divide by              | 4)0.0253059(0.0063264.75   |
| Log. of the first mean | 2.0000000                  |
| Excess added           | 0.0063264.75               |
| Log. of the first mean | 101.4673846 = 2.0063264.75 |
| Excess added           | 0.0063264.75               |
| Log. of the second     | 102.9563014 = 2.0126529.50 |
| Excess added           | 0.0063264.75               |
| Log. of the third      | 104.4670483 = 2.0189794.25 |
| Excess added           | 0.0063264.75               |
| Log. of the fourth     | 106 — 2.0253059            |

*The use of the tables of Logarithmic sines, tangents and secants.*

1. To find the sine, tangent and secant of any arch to ninety degrees.

↓

Find



Find the number of degrees at the head or foot of the table; and the minutes in the left or right hand column, and in the common angle of meeting, formed by the title placed at the head or foot of the column, and the minutes, you will find the sine, tangent or secant required.

But if the given arch contains any parts of a minute, intermediate to those found in the tables; take the difference between the sines, &c. of the given degrees and minutes, and of the minute next greater.

Then as 1 minute is to that difference, so is the given intermediate part of a minute in decimals to a fourth proportional, Which being added to the sine, &c. first found will give the sine, &c. required.

*Example.* The Log. sine of  $1^{\circ}.48'.28''.12'''$ , is required.

|                                     |           |
|-------------------------------------|-----------|
| The Log. sine of $1^{\circ}.49'$ is | 8.5010798 |
| The Log. sine of $1^{\circ}.48'$ is | 8.4970784 |
| Their difference is                 | 40014     |

Then as  $1 : 40014 :: 0'.47 (=28''.12''') : 18807$  nearly.

|                                                        |           |
|--------------------------------------------------------|-----------|
| Therefore to the Log. sine of $1^{\circ}.48'$          | 8.4970784 |
| Add the difference found                               | 40014     |
| The sum is the Log. sine of $1^{\circ}.48'.28''.12'''$ | 8.5010798 |

2. To find the arch answering to any given sine, tangent, or secant.

Take the next less found in the tables, and subtract it from that given, observing the degrees and minutes answering to it, and divide this remainder by the difference between it and the next greater, adding as many cyphers as are necessary to the dividend, and the quotient will be the decimal part of a minute to be added to the degrees and minutes before found.

*Example 1.* Suppose it were required to find the arch answering to the Log. sine

|                                                    |           |
|----------------------------------------------------|-----------|
| The next less is the Log. sine of $43^{\circ}.41'$ | 9.8392719 |
| The difference is                                  | 1140      |

Also the difference between the Log. sine of  $43^{\circ}.41'$ , and  $43^{\circ}.42'$  is 1322.

Then  $1322)1140.000(0'.862 = 51''.43'''$ , which being added to  $43^{\circ}.41'$ , gives  $43^{\circ}.41'.51''.43'''$ , for the arch required.

*Example*

# AN APPENDIX.

*Example 2.* Let it be required to find the arch corresponding to the Log. tangent 9.6766687  
 The tang. of  $25^{\circ}.24'$ , the next less is 9.6765426  
 The difference is 1261

Also the difference between the Log. tangent of  $25^{\circ}.24'$  and the Log. tangent of  $25^{\circ}.25'$  is 3260.

Then,  $3260)1261.0000(0'.387 = 23''.13'''$ .  
 Therefore  $25^{\circ}.24'.23''.13'''$  is the arch answering to the given tangent.

**F I N I S.**



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